Unified Number Theoretical Vision

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Contents

Abstract

An updated view about M^8-H duality is discussed. M^8-H duality allows to deduce $M^4\times CP_2$ via number theoretical compactification. One important correction is that octonionic spinor structure makes sense only for M^8 whereas for $M^4 \times CP_2$ complefixied quaternions characterized the spinor structure.

Octonions, quaternions associative and co-associative space-time surfaces, octonionic spinors and twistors and twistor spaces are highly relevant for quantum TGD. In the following some general observations distilled during years are summarized.

There is a beautiful pattern present suggesting that $H = M^4\times CP_2$ is completely unique on number theoretical grounds. Consider only the following facts. M^4 and CP_2 are the unique 4-D spaces allowing twistor space with Kähler structure. Octonionic projective space OP_2 appears as octonionic twistor space (there are no higher-dimensional octonionic projective spaces). Octotwistors generalise the twistorial construction from M^4 to M^8 and octonionic gamma matrices make sense also for H with quaternionicity condition reducing OP_2 to to 12-D $G_2/U(1) \times U(1)$ having same dimension as the the twistor space $CP_3 \times SU(3)/U(1) \times U(1)$ of H assignable to complexified quaternionic representation of gamma matrices.

A further fascinating structure related to octo-twistors is the non-associated analog of Lie group defined by automorphisms by octonionic imaginary units: this group is topologically six-sphere. Also the analogy of quaternionicity of preferred extremals in TGD with the Majorana condition central in super string models is very thought provoking. All this suggests that associativity indeed could define basic dynamical principle of TGD.

Number theoretical vision about quantum TGD involves both p-adic number fields and classical number fields and the challenge is to unify these approaches. The challenge is non-trivial since the p-adic variants of quaternions and octonions are not number fields without additional conditions. The key idea is that TGD reduces to the representations of Galois group of algebraic numbers realized in the spaces of octonionic and quaternionic adeles generalizing the ordinary adeles as Cartesian products of all number fields: this picture relates closely to Langlands program. Associativity would force sub-algebras of the octonionic adeles defining 4-D surfaces in the space of octonionic adeles so that 4-D space-time would emerge naturally. $M^8 - H$ correspondence in turn would map the space-time surface in M^8 to $M^4 \times CP_2$.

A long-standing question has been the origin of preferred p-adic primes characterizing elementary particles. I have proposed several explanations and the most convincing hitherto is related to the algebraic extensions of rationals and p-adic numbers selecting naturally preferred primes as those which are ramified for the extension in question.

1 Introduction

Octonions, quaternions, quaternionic space-time surfaces, octonionic spinors and twistors and twistor spaces are highly relevant for quantum TGD. In the following some general observations distilled during years are summarized. This summary involves several corrections to the picture which has been developing for a decade or so.

A brief updated view about $M^8 - H$ duality and twistorialization is in order. There is a beautiful pattern present suggesting that $M^8 - H$ duality makes sense and that $H = M^4 \times CP_2$ is completely unique on number theoretical grounds.

1. M^8-H duality allows to deduce $M^4\times CP_2$ via number theoretical compactification. For the option with minimal number of conjectures the associativity/co-associativity of the space-time surfaces in M^8 guarantees that the space-time surfaces in M^8 define space-time surfaces in H. The tangent/normal spaces of quaternionic/hyper-quaternionic surfaces in M^8 contain also an integrable distribution of hyper-complex tangent planes $M^2(x)$.

   An important correction is that associativity/co-associativity does not make sense at the level of H since the spinor structure of H is already complex quaternionic and reducible to the ordinary one by using matrix representations for quaternions. The associativity condition should however have some counterpart at level of H. One could require that the induced gamma matrices at each point could span a *real*-quaternionic sub-space of complexified quaternions for quaternionicity and a purely imaginary quaternionic sub-space for co-quaternionicity. One might hope that it is consistent with - or even better, implies - preferred extremal property. I have not however found a viable definition of quaternionic "reality". On the other hand, it is possible assigne the tangent space M^8 of H with octonion structure and define associativity as in case of M^8.

$M^8 - H$ duality could generalize to $H - H$ duality in the sense that also the image of the space-time surface under duality map is not only preferred extremal but also associative (co-associative) surface. The duality map $H \to H$ could be iterated and would define the arrow for the category formed by preferred extremals.

2. M^4 and CP_2 are the unique 4-D spaces allowing twistor space with Kähler structure. M^8 allows twistor space for octonionic spinor structure obtained by direct generalization of the standard construction for M^4. $M^4 \times CP_2$ spinors can be regarded as tensor products of quaternionic spinors associated with M^4 and CP_2: this trivial observation forces to challenge the earlier rough vision, which however seems to stand up the challenge.

3. Octotwistors generalise the twistorial construction from M^4 to M^8 and octonionic gamma matrices make sense also for H with quaternionicity condition reducing 12-D $T(M^8) = G_2/U(1)\times U(1)$ to the 12-D twistor space $T(H) = CP_3 \times SU^3/U(1)\times U(1)$. The interpretation of the twistor space in the case of M^8 is as the space of choices of quantization axes for the 2-D Cartan algebra of G_2 acting as octonionic automorphisms. For CP_2 one has space for the chocies of quantization axes for the 2-D $SU(3)$ Cartan algebra.

4. It is also possible that the dualities extend to a sequence $M^8 \to H \to H$... by mapping the associative/co-associative tangent space to CP_2 and M^4 point to M^4 point at each step. One has good reasons to expect that this iteration generates fractal as the limiting space-time surface.

5. A fascinating structure related to octo-twistors is the non-associated analog of Lie group defined by automorphisms by octonionic imaginary units: this group is topologically 7-sphere. Second analogous structure is the 7-D Lie algebra like structure defined by octonionic analogs of sigma matrices.

The analogy of quaternionicity of M^8 pre-images of preferred extremals and quaternionicity of the tangent space of space-time surfaces in H with the Majorana condition central in super string models is very thought provoking. All this suggests that associativity at the level of M^8 indeed could define basic dynamical principle of TGD.

Number theoretical vision about quantum TGD involves both p-adic number fields and classical number fields and the challenge is to unify these approaches. The challenge is non-trivial since the p-adic variants of quaternions and octonions are not number fields without additional conditions. The key idea is that TGD reduces to the representations of Galois group of algebraic numbers realized in the spaces of octonionic and quaternionic adeles generalizing the ordinary adeles as Cartesian products of all number fields: this picture relates closely to Langlands program. Associativity would force sub-algebras of the octonionic adeles defining 4-D surfaces in the space of octonionic adeles so that 4-D space-time would emerge naturally. $M^8 - H$ correspondence in turn would map the space-time surface in M^8 to $M^4 \times CP_2$.

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L2].

2 Number Theoretic Compactification And $M^8 - H$ Duality

This section summarizes the basic vision about number theoretic compactification reducing the classical dynamics to associativity or co-associativity. Originally $M^8 - H$ duality was introduced as a number theoretic explanation for $H = M^4 \times CP_2$. Much later it turned out that the completely exceptional twistorial properties of M^4 and CP_2 are enough to justify $X^4 \subset H$ hypothesis. Skeptic could therefore criticize the introduction of M^8 (actually its complexification) as an un-necessary mathematical complication producing only unproven conjectures and bundle of new statements to be formulated precisely. However, if quaternionicity can be realized in terms of M^8_c using O_c-real analytic functions and if quaternionicity is equivalent with preferred extremal property, a huge simplification results and one can say that field equations are exactly solvable.

One can question the feasibility of $M^8 - H$ duality if the dynamics is purely number theoretic at the level of M^8 and determined by Kähler action at the level of H. Situation becomes more democratic if Kähler action defines the dynamics in both M^8 and H: this might mean that associativity could imply field equations for preferred extremals or vice versa or there might be equivalence between two. This means the introduction Kähler structure at the level of M^8, and motivates also the coupling of Kähler gauge potential to M^8 spinors characterized by Kähler charge or em charge. One could call this form of duality strong form of $M^8 - H$ duality.

The strong form $M^8 - H$ duality boils down to the assumption that space-time surfaces can be regarded either as 4-surfaces of H or as surfaces of M^8 or even M^8_c composed of associative and co-associative regions identifiable as regions of space-time possessing Minkowskian *resp.* Euclidian signature of the induced metric. They have the same induced metric and Kähler form and WCW associated with H should be essentially the same as that associated with M^8. Associativity corresponds to hyper-quaterniocity at the level of tangent space and co-associativity to co-hyper-quaternionicity - that is associativity/hyper-quaternionicity of the normal space. Both are needed to cope with known extremals. Since in Minkowskian context precise language would force to introduce clumsy terms like hyper-quaternionicity and co-hyper-quaternionicity, it is better to speak just about associativity or co-associativity.

Remark: The original assumption was that space-times could be regarded as surfaces in M^8 rather than in its complexification M^8_c identifiable as complexified octonions. This assumption is un-necessarily strong and if one assumes that octonion-real analytic functions characterize these surfaces M^8_c must be assumed.

For the octonionic spinor fields the octonionic analogs of electroweak couplings reduce to mere K̈hler or electromagnetic coupling and the solutions reduce to those for spinor d'Alembertian in 4-D harmonic potential breaking $SO(4)$ symmetry. Due to the enhanced symmetry of harmonic oscillator, one expects that partial waves are classified by $SU(4)$ and by reduction to $SU(3) \times U(1)$ by em charge and color quantum numbers just as for CP_2 - at least formally.

Harmonic oscillator potential defined by self-dual em field splits M^8 to $M^4 \times E^4$ and implies Gaussian localization of the spinor modes near origin so that E^4 effectively compactifies. The The resulting physics brings strongly in mind low energy physics, where only electromagnetic interaction is visible directly, and one cannot avoid associations with low energy hadron physics. These are some of the reasons for considering $M^8 - H$ duality as something more than a mere mathematical curiosity.

Remark: The Minkowskian signatures of M^8 and M^4 produce technical nuisance. One could overcome them by Wick rotation, which is however somewhat questionable trick. $M^8_c = O_c$ provides the proper formulation.

1. The proper formulation is in terms of complexified octonions and quaternions involving the introduction of commuting imaginary unit j.

2. Hyper-quaternions/octonions define as subspace of complexified quaternions/octonions spanned by real unit and jI_k, where I_k are quaternionic units. These spaces are obviously not closed under multiplication. One can however however define the notion of associativity for the subspace of M^8 by requiring that the products and sums of the tangent space vectors generate complexified quaternions.

3. Ordinary quaternions Q are expressible as $q = q_0 + q^k I_k$. Hyper-quaternions are expressible as $q = q_0 + jq^k I_k$ and form a subspace of complexified quaternions $Q_c = Q \oplus jQ$. Similar formula applies to octonions and their hyper counterparts which can be regarded as subspaces of complexified octonions $O \oplus jO$. Tangent space vectors of H correspond hyper-quaternions $q_H = q_0 + jq^k I_k + jiq_2$ defining a subspace of doubly complexified quaternions: note the appearance of two imaginary units.

The recent definitions of associativity and M^8 duality has evolved slowly from in-accurate characterizations and there are still open questions.

1. Kähler form for M^8 non-trivial only in $E^4 \subset M^8$ implies unique decomposition $M^8 = M^4 \times E^4$ needed to define $M^8 - H$ duality uniquely. This applies also to M^8_c. This forces to introduce also Kähler action, induced metric and induced Kähler form. Could strong form of duality

meant that the space-time surfaces in M^8 and H have same induced metric and induced Kähler form? Could the WCW s associated with M^8 and H be identical with this assumption so that duality would provide different interpretations for the same physics?

2. One can formulate associativity in M^8 (or M^8_c) by introducing octonionic structure in tangent spaces or in terms of the octonionic representation for the induced gamma matrices. Does the notion have counterpart at the level of H as one might expect if Kähler action is involved in both cases? The analog of this formulation in H might be as quaternionic "reality" since tangent space of H corresponds to complexified quaternions: I have however found no acceptable definition for this notion.

   The earlier formulation is in terms of octonionic flat space gamma matrices replacing the ordinary gamma matrices so that the formulation reduces to that in M^8 tangent space. This formulation is enough to define what associativity means although one can protest. Somehow H is already complex quaternionic and thus associative. Perhaps this just what is needed since dynamics has two levels: *embedding space level* and *space-time level*. One must have embedding space spinor harmonics assignable to the ground states of super-conformal representations and quaternionicity and octonionicity of H tangent space would make sense at the level of space-time surfaces.

3. Whether the associativity using induced gamma matrices works is not clear for massless extremals (MEs) and vacuum extremals with the dimension of CP_2 projection not larger than 2.

4. What makes this notion of associativity so fascinating is that it would allow to iterate duality as a sequence $M^8 \rightarrow H \rightarrow H...$ by mapping the space-time surface to $M^4 \times CP_2$ by the same recipe as in case of M^8. This brings in mind the functional composition of O_c-real analytic functions (O_c denotes complexified octonions: complexification is forced by Minkowskian signature) suggested to produced associative or co-associative surfaces. The associative (co-associative) surfaces in M^8 would correspond to loci for vanishing of imaginary (real) part of octonion-real-analytic function.

It might be possible to define associativity in H also in terms of Kähler-Dirac gamma matrices defined by Kähler action (certainly not M^8).

1. All known extremals are associative or co-associative in H in this sense. This would also give direct correlation with the variational principle. For the known preferred extremals this variant is successful partially because the Kähler-Dirac gamma matrices need not span the entire tangent space. The space spanned by the Kähler-Dirac gammas is not necessarily tangent space. For instance for CP_2 type vacuum extremals the Kähler-Dirac gamma matrices are CP_2 gamma matrices plus an additional light-like component from M^4 gamma matrices.

   If the space spanned by Kähler-Dirac gammas has dimension D smaller than 3 co-associativity is automatic. If the dimension of this space is $D = 3$ it can happen that the triplet of gammas spans by multiplication entire octonionic algebra. For $D = 4$ the situation is of course non-trivial.

2. For Kähler-Dirac gamma matrices the notion of co-associativity can produce problems since Kähler-Dirac gamma matrices do not in general span the tangent space. What does co-associativity mean now? Should one replace normal space with orthogonal complement of the space spanned by Kähler-Dirac gamma matrices? Co-associativity option must be considered for $D = 4$ only. CP_2 type vacuum extremals provide a good example. In this case the Kähler-Dirac gamma matrices reduce to sums of ordinary CP_2 gamma matrices and ligt-like M^4 contribution. The orthogonal complement for the Kähler-Dirac gamma matrices consists of dual light-like gamma matrix and two gammas orthogonal to it: this space is subspace of M^4 and trivially associative.

2.1 Basic Idea Behind $M^8 - M^4 \times CP_2$ Duality

If four-surfaces $X^4 \subset M^8$ under some conditions define 4-surfaces in $M^4 \times CP_2$ indirectly, the spontaneous compactification of super string models would correspond in TGD to two different ways to interpret the space-time surface. This correspondence could be called number theoretical compactification or $M^8 - H$ duality.

The hard mathematical facts behind the notion of number theoretical compactification are following.

1. One must assume that M^8 has unique decomposition $M^8 = M^4 \times E^4$. This decomposition generalizes also to the case of M^8_c. This would be most naturally due to Kähler structure in E^4 defined by a self-dual Kähler form defining parallel constant electric and magnetic fields in Euclidian sense. Besides Kähler form there is vector field coupling to sigma matrix representing the analog of strong isospin: the corresponding octonionic sigma matrix however is imaginary unit times gamma matrix - say ie_1 in M^4 - defining a preferred plane M^2 in M^4. Here it is essential that the gamma matrices of E^4 defined in terms of octonion units commute to gamma matrices in M^4. What is involved becomes clear from the Fano triangle illustrating octonionic multiplication table.

2. The space of hyper-complex structures of the hyper-octonion space - they correspond to the choices of plane $M^2 \subset M^8$ - is parameterized by 6-sphere $S^6 = G^2/SU(3)$. The subgroup $SU(3)$ of the full automorphism group G_2 respects the a priori selected complex structure and thus leaves invariant one octonionic imaginary unit, call it e_1. Fixed complex structure therefore corresponds to a point of S^6.

3. Quaternionic sub-algebras of M^8 (and M^8_c) are parametrized by $G_2/U(2)$. The quaternionic sub-algebras of octonions with fixed complex structure (that is complex sub-space defined by real and preferred imaginary unit and parametrized by a point of S^6) are parameterized by $SU(3)/U(2) = CP_2$ just as the complex planes of quaternion space are parameterized by $CP_1 = S^2$. Same applies to hyper-quaternionic sub-spaces of hyper-octonions. $SU(3)$ would thus have an interpretation as the isometry group of CP_2, as the automorphism sub-group of octonions, and as color group. Thus the space of quaternionic structures can be parametrized by the 10-dimensional space $G_2/U(2)$ decomposing as $S^6 \times CP_2$ locally.

4. The basic result behind number theoretic compactification and $M^8 - H$ duality is that associative sub-spaces $M^4 \subset M^8$ containing a fixed commutative sub-space $M^2 \subset M^8$ are parameterized by CP_2. The choices of a fixed hyper-quaternionic basis $1, e_1, e_2, e_3$ with a fixed complex sub-space (choice of e_1) are labeled by $U(2) \subset SU(3)$. The choice of e_2 and e_3 amounts to fixing $e_2 \pm \sqrt{-1}e_3$, which selects the $U(2) = SU(2) \times U(1)$ subgroup of $SU(3)$. $U(1)$ leaves 1 invariant and induced a phase multiplication of e_1 and $e_2 \pm e_3$. $SU(2)$ induces rotations of the spinor having e_2 and e_3 components. Hence all possible completions of $1, e_1$ by adding e_2, e_3 doublet are labeled by $SU(3)/U(2) = CP_2$.

Consider now the formulation of $M^8 - H$ duality.

1. The idea of the standard formulation is that associative manifold $X^4 \subset M^8$ has at its each point associative tangent plane. That is X^4 corresponds to an integrable distribution of $M^2(x) \subset M^8$ parametrized 4-D coordinate x that is map $x \to S^6$ such that the 4-D tangent plane is hyper-quaternionic for each x.

2. Since the Kähler structure of M^8 implies unique decomposition $M^8 = M^4 \times E^4$, this surface in turn defines a surface in $M^4 \times CP_2$ obtained by assigning to the point of 4-surface point $(m, s) \in H = M^4 \times CP_2$: $m \in M^4$ is obtained as *projection* $M^8 \to M^4$ (this is modification to the earlier definition) and $s \in CP_2$ parametrizes the quaternionic tangent plane as point of CP_2. Here the local decomposition $G_2/U(2) = S^6 \times CP_2$ is essential for achieving uniqueness.

3. One could also map the associative surface in M^8 to surface in 10-dimensional $S^6 \times CP_2$. In this case the metric of the image surface cannot have Minkowskian signature and one cannot assume that the induced metrics are identical. It is not known whether S^6 allows genuine complex structure and Kähler structure which is essential for TGD formulation.

4. Does duality imply the analog of associativity for $X^4 \subset H$? The tangent space of H can be seen as a sub-space of doubly complexified quaternions. Could one think that quaternionic

sub-space is replaced with sub-space analogous to that spanned by real parts of complexified quaternions? The attempts to define this notion do not however look promising. One can however define associativity and co-associativity for the tangent space M^8 of H using octonionization and can formulate it also terms of induced gamma matrices.

5. The associativity defined in terms of induced gamma matrices in both in M^8 and H has the interesting feature that one can assign to the associative surface in H a new associative surface in H by assigning to each point of the space-time surface its M^4 projection and point of CP_2 characterizing its associative tangent space or co-associative normal space. It seems that one continue this series ad infinitum and generate new solutions of field equations! This brings in mind iteration which is standard manner to generate fractals as limiting sets. This certainly makes the heart of mathematician beat.

6. Kähler structure in $E^4 \subset M^8$ guarantees natural $M^4 \times E^4$ decomposition. Does associativity imply preferred extremal property or vice versa, or are the two notions equivalent or only consistent with each other for preferred extremals?

A couple of comments are in order.

1. This definition generalizes to the case of M_c^8: all that matters is that tangent space-is is complexified quaternionic and there is a unique identification $M^4 \subset M_c^8$: this allows to assign the point of 4-surfaces a point of $M^4 \times CP_2$. The generalization is needed if one wants to formulate the hypothesis about O_c real-analyticity as a way to build quaternionic space-time surfaces properly.

2. This definition differs from the first proposal for years ago stating that each point of X^4 contains a *fixed* $M^2 \subset M^4$ rather than $M_2(x) \subset M^8$ and also from the proposal assuming integrable distribution of $M^2(x) \subset M^4$. The older proposals are not consistent with the properties of massless extremals and string like objects for which the counterpart of M^2 depends on space-time point and is not restricted to M^4. The earlier definition $M^2(x) \subset M^4$ was problematic in the co-associative case since for the Euclidian signature is is not clear what the counterpart of $M^2(x)$ could be.

3. The new definition is consistent with the existence of Hamilton-Jacobi structure meaning slicing of space-time surface by string world sheets and partonic 2-surfaces with points of partonic 2-surfaces labeling the string world sheets [K6]. This structure has been proposed to characterize preferred extremals in Minkowskian space-time regions at least.

4. Co-associative Euclidian 4-surfaces, say CP_2 type vacuum extremal do not contain integrable distribution of $M^2(x)$. It is normal space which contains $M^2(x)$. Does this have some physical meaning? Or does the surface defined by $M^2(x)$ have Euclidian analog?

   A possible identification of the analog would be as string world sheet at which W boson field is pure gauge so that the modes of the modified Dirac operator [K45] restricted to the string world sheet have well-defined em charge. This condition appears in the construction of solutions of Kähler-Dirac operator.

   For octonionic spinor structure the W coupling is however absent so that the condition does not make sense in M^8. The number theoretic condition would be as commutative or co-commutative surface for which imaginary units in tangent space transform to real and imaginary unit by a multiplication with a fixed imaginary unit! One can also formulate co-associativity as a condition that tangent space becomes associative by a multiplication with a fixed imaginary unit.

   There is also another justification for the distribution of Euclidian tangent planes. The idea about associativity as a fundamental dynamical principle can be strengthened to the statement that space-time surface allows slicing by hyper-complex or complex 2-surfaces, which are commutative or co-commutative inside space-time surface. The physical interpretation would be as Minkowskian or Euclidian string world sheets carrying spinor modes. This would give a connection with string model and also with the conjecture about the general structure of preferred extremals.

5. Minimalist could argue that the minimal definition requires octonionic structure and associativity *only* in M^8. There is no need to introduce the counterpart of Kähler action in M^8 since the dynamics would be based on associativity or co-associativity alone. The objection is that one must assumes the decomposition $M^8 = M^4 \times E^4$ without any justification.

   The map of space-time surfaces to those of $H = M^4 \times CP_2$ implies that the space-time surfaces in H are in well-defined sense quaternionic. As a matter of fact, the standard spinor structure of H can be regarded as quaternionic in the sense that gamma matrices are essentially tensor products of quaternionic gamma matrices and reduce in matrix representation for quaternions to ordinary gamma matrices. Therefore the idea that one should introduce octonionic gamma matrices in H is questionable. If all goes as in dreams, the mere associativity or co-associativity would code for the preferred extremal property of Kähler action in H. One could at least hope that associativity/co-associativity in H is consistent with the preferred extremal property.

6. One can also consider a variant of associativity based on modified gamma matrices - but only in H. This notion does not make sense in M^8 since the very existence of quaternionic tangent plane makes it possible to define $M^8 - H$ duality map. The associativity for modified gamma matrices is however consistent with what is known about extremals of Kähler action. The associativity based on induced gamma matrices would correspond to the use of the space-time volume as action. Note however that gamma matrices are *not* necessary in the definition.

2.2 Hyper-Octonionic Pauli "Matrices" And The Definition Of Associativity

Octonionic Pauli matrices suggest an interesting possibility to define precisely what associativity means at the level of M^8 using gamma matrices (for background see [K39, K4]).

1. According to the standard definition space-time surface $X^4 \subset M^8$ is associative if the tangent space at each point of X^4 in $X^4 \subset M^8$ picture is associative. The definition can be given also in terms of octonionic gamma matrices whose definition is completely straightforward.

2. Could/should one define the analog of associativity at the level of H? One can identify the tangent space of H as M^8 and can define octonionic structure in the tangent space and this allows to define associativity locally. One can replace gamma matrices with their octonionic variants and formulate associativity in terms of them locally and this should be enough.

   Skeptic however reminds M^4 allows hyper-quaternionic structure and CP_2 quaternionic structure so that complexified quaternionic structure would look more natural for H. The tangent space would decompose as $M^8 = HQ + ijQ$, weher j is commuting imaginary unit and HQ is spanned by real unit and by units iI_k, where i second commutating imaginary unit and I_k denotes quaternionic imaginary units. There is no need to make anything associative.

   There is however far from obvious that octonionic spinor structure can be (or need to be!) defined globally. The lift of the CP_2 spinor connection to its octonionic variant has questionable features: in particular vanishing of the charged part and reduction of neutral part to photon. Therefore is is unclear whether associativity condition makes sense for $X^4 \subset M^4 \times CP_2$. What makes it so fascinating is that it would allow to iterate duality as a sequences $M^8 \rightarrow H \rightarrow H....$ This brings in mind the functional composition of octonion real-analytic functions suggested to produced associative or co-associative surfaces.

I have not been able to settle the situation. What seems the working option is associativity in both M^8 and H and Kähler-Dirac gamma matrices defined by appropriate Kähler action and correlation between associativity and preferred extremal property.

2.3 Are Kähler And Spinor Structures Necessary In M^8?

If one introduces M^8 as dual of H, one cannot avoid the idea that hyper-quaternionic surfaces obtained as images of the preferred extremals of Kähler action in H are also extremals of M^8 Kähler action with same value of Kähler action defining Kähler function. As found, this leads to the conclusion that the $M^8 - H$ duality is Kähler isometry. Coupling of spinors to Kähler potential is the next step and this in turn leads to the introduction of spinor structure so that quantum TGD in H should have full M^8 dual.

2.3.1 Are also the 4-surfaces in M^8 preferred extremals of Kähler action?

It would be a mathematical miracle if associative and co-associative surfaces in M^8 would be in 1-1 correspondence with preferred extremals of Kähler action. This motivates the question whether Kähler action make sense also in M^8. This does not exclude the possibility that associativity implies or is equivalent with the preferred extremal property.

One expects a close correspondence between preferred extremals: also now vacuum degeneracy is obtained, one obtains massless extremals, string like objects, and counterparts of CP_2 type vacuum extremals. All known extremals would be associative or co-associative if modified gamma matrices define the notion (possible only in the case of H).

The strongest form of duality would be that the space-time surfaces in M^8 and H have same induced metric same induced Kähler form. The basic difference would be that the spinor connection for surfaces in M^8 would be however neutral and have no left handed components and only em gauge potential. A possible interpretation is that M^8 picture defines a theory in the phase in which electroweak symmetry breaking has happened and only photon belongs to the spectrum.

The question is whether one can define WCW also for M^8. Certainly it should be equivalent with WCW for H: otherwise an inflation of poorly defined notions follows. Certainly the general formulation of the WCW geometry generalizes from H to M^8. Since the matrix elements of symplectic super-Hamiltonians defining WCW gamma matrices are well defined as matrix elements involve spinor modes with Gaussian harmonic oscillator behavior, the non-compactness of E^4 does not pose any technical problems.

2.3.2 Spinor connection of M^8

There are strong physical constraints on M^8 dual and they could kill the hypothesis. The basic constraint to the spinor structure of M^8 is that it reproduces basic facts about electro-weak interactions. This includes neutral electro-weak couplings to quarks and leptons identified as different H-chiralities and parity breaking.

1. By the flatness of the metric of E^4 its spinor connection is trivial. E^4 however allows full S^2 of covariantly constant Kähler forms so that one can accommodate free independent Abelian gauge fields assuming that the independent gauge fields are orthogonal to each other when interpreted as realizations of quaternionic imaginary units. This is possible but perhaps a more natural option is the introduction of just single Kähler form as in the case of CP_2.

2. One should be able to distinguish between quarks and leptons also in M^8, which suggests that one introduce spinor structure and Kähler structure in E^4. The Kähler structure of E^4 is unique apart form $SO(3)$ rotation since all three quaternionic imaginary units and the unit vectors formed from them allow a representation as an antisymmetric tensor. Hence one must select one preferred Kähler structure, that is fix a point of S^2 representing the selected imaginary unit. It is natural to assume different couplings of the Kähler gauge potential to spinor chiralities representing quarks and leptons: these couplings can be assumed to be same as in case of H.

3. Electro-weak gauge potential has vectorial and axial parts. Em part is vectorial involving coupling to Kähler form and Z^0 contains both axial and vector parts. The naïve replacement of sigma matrices appearing in the coupling of electroweak gauge fields takes the left handed parts of these fields to zero so that only neutral part remains. Further, gauge fields correspond to curvature of CP_2 which vanishes for E^4 so that only Kähler form form remains. Kähler form couples to 3L and q so that the basic asymmetry between leptons and quarks remains. The resulting field could be seen as analog of photon.

4. The absence of weak parts of classical electro-weak gauge fields would conform with the standard thinking that classical weak fields are not important in long scales. A further prediction is that this distinction becomes visible only in situations, where H picture is

necessary. This is the case at high energies, where the description of quarks in terms of $SU(3)$ color is convenient whereas $SO(4)$ QCD would require large number of E^4 partial waves. At low energies large number of $SU(3)$ color partial waves are needed and the convenient description would be in terms of $SO(4)$ QCD. Proton spin crisis might relate to this.

2.3.3 Dirac equation for leptons and quarks in M^8

Kähler gauge potential would also couple to octonionic spinors and explain the distinction between quarks and leptons.

1. The complexified octonions representing H spinors decompose to $1+1+3+\overline{3}$ under SU(3) representing color automorphisms but the interpretation in terms of QCD color does not make sense. Rather, the triplet and single combine to two weak isospin doublets and quarks and leptons corresponds to "spin" states of octonion valued 2-spinor. The conservation of quark and lepton numbers follows from the absence of coupling between these states.

2. One could modify the coupling so that coupling is on electric charge by coupling it to electromagnetic charge which as a combination of unit matrix and sigma matrix is proportional to $1+kI_1$, where I_1 is octonionic imaginary unit in $M^2 \subset M^4$. The complexified octonionic units can be chosen to be eigenstates of Q_{em} so that Laplace equation reduces to ordinary scalar Laplacian with coupling to self-dual em field.

3. One expects harmonic oscillator like behavior for the modes of the Dirac operator of M^8 since the gauge potential is linear in E^4 coordinates. One possibility is Cartesian coordinates is $A(A_x, A_y, A_z, A_t) = k(-y, x, t, -z)$. Thhe coupling would make E^4 effectively a compact space.

4. The square of Dirac operator gives potential term proportional to $r^2 = x^2+y^2+z^2+t^2$ so that the spectrum of 4-D harmonic oscillator operator and $SO(4)$ harmonics localized near origin are expected. For harmonic oscillator the symmetry enhances to $SU(4)$.

   If one replaces Kähler coupling with em charge symmetry breaking of $SO(4)$ to vectorial $SO(3)$ is expected since the coupling is proportional to $1+ike_1$ defining electromagnetic charge. Since the basis of complexified quaternions can be chosen to be eigenstates of e_1 under multiplication, octonionic spinors are eigenstates of em charge and one obtains two color singles $1 \pm e_1$ and color triplet and antitriplet. The color triplets cannot be however interpreted in terms of quark color.

   Harmonic oscillator potential is expected to enhance $SO(3)$ to SU(3). This suggests the reduction of the symmetry to $SU(3) \times U(1)$ corresponding to color symmetry and em charge so that one would have same basic quantum numbers as tof CP_2 harmonics. An interesting question is how the spectrum and mass squared eigenvalues of harmonics differ from those for CP_2.

5. In the square of Dirac equation $J^{kl}\Sigma_{kl}$ term distinguishes between different em charges ($\Sigma_k l$ reduces by self duality and by special properties of octonionic sigma matrices to a term proportional to iI_1 and complexified octonionic units can be chosen to be its eigenstates with eigen value ± 1. The vacuum mass squared analogous to the vacuum energy of harmonic oscillator is also present and this contribution are expected to cancel themselves for neutrinos so that they are massless whereas charged leptons and quarks are massive. It remains to be checked that quarks and leptons can be classified to triality $T = \pm 1$ and $t = 0$ representations of dynamical $SU(3)$ respectively.

2.3.4 What about the analog of Kähler Dirac equation

Only the octonionic structure in $T(M^8)$ is needed to formulate quaternionicity of space-time surfaces: the reduction to O_c-real-analyticity would be extremely nice but not necessary (O_c denotes complexified octonions needed to cope with Minkowskian signature). Most importantly, there might be no need to introduce Kähler action (and Kähler form) in M^8. Even the octonionic representation of gamma matrices is un-necessary. Neither there is any absolute need to define octonionic Dirac equation and octonionic Kähler Dirac equation nor octonionic analog of its solutions nor the octonionic variants of embedding space harmonics.

It would be of course nice if the general formulas for solutions of the Kähler Dirac equation in H could have counterparts for octonionic spinors satisfying quaternionicity condition. One can indeed wonder whether the restriction of the modes of induced spinor field to string world sheets defined by integrable distributions of hyper-complex spaces $M^2(x)$ could be interpretated in terms of commutativity of fermionic physics in M^8. $M^8 - H$ correspondence could map the octonionic spinor fields at string world sheets to their quaternionic counterparts in H. The fact that only holomorphy is involved with the definition of modes could make this map possible.

2.4 How Could One Solve Associativity/Co-Associativity Conditions?

The natural question is whether and how one could solve the associativity/-co-associativity conditions explicitly. One can imagine two approaches besides $M^8 \rightarrow H \rightarrow H...$ iteration generating new solutions from existing ones.

2.4.1 Could octonion-real analyticity be equivalent with associativity/co-associativity?

Analytic functions provide solutions to 2-D Laplace equations and one might hope that also the field equations could be solved in terms of octonion-real-analyticity at the level of M^8 perhaps also at the level of H. Signature however causes problems - at least technical. Also the compactness of CP_2 causes technical difficulties but they need not be insurmountable.

For E^8 the tangent space would be genuinely octonionic and one can define the notion octonion-real analytic map as a generalization of real-analytic function of complex variables (the coefficients of Laurent series are real to guarantee associativity of the series). The argument is complexified octonion in $O\oplus iO$ forming an algebra but not a field. The norm square is Minkowskian as difference of two Euclidian octonionic norms: $N(o_1 + io_2) = N(o_1) - N(o_2)$ and vanishes at 15-D light cone boundary. Obviously, differential calculus is possible outside the light-cone boundary. Rational analytic functions have however poles at the light-cone boundary. One can wonder whether the poles at M^4 light-cone boundary, which is subset of 15-D light-cone boundary could have physical significance and relevant for the role of causal diamonds in ZEO.

The candidates for associative surfaces defined by O_c-real-analytic functions (I use O_c for complexified octonions) have Minkowskian signature of metric and are 4-surfaces at which the projection of $f(o_1 + io_2)$ to $Im(O_1)$, $iIm(O_2)$, and $iRe(Q_2) \oplus Im(Q_1)$ vanish so that only the projection to hyper-quaternionic Minkowskian sub-space $M^4 = Re(Q_1) + iIm(Q_2)$ with signature $(1, -1, -, 1-, 1)$ is non-vanishing. The inverse image need not belong to M^8 and in general it belongs to M^8_c but this is not a problem: all that is needed that the tangent space of inverse image is complexified quaternionic. If this is the case then $M^8 - H$ duality maps the tangent space of the inverse image to CP_2 point and image itself defines the point of M^4 so that a point of H is obtained. Co-associative surfaces would be surfaces for which the projections of image to $Re(O_1)$, $iRe(O_2)$, and to $Im(O_1)$ vanish so that only the projection to $iIm(O_2)$ with signature $(-1, -1, -1, -1)$ is non-vanishing.

The inverse images as 4-D sub-manifolds of M^8_c (not M^8!) are excellent candidates for associative and co-associative 4-surfaces since $M^8 - H$ duality assignes to them a 4-surface in $M^4 \times CP_2$ if the tangent space at given point is complexified quaternionic. This is true if one believes on the analytic continuation of the intuition from complex analysis (the image of real axes under the map defined by O_c-real-analytic function is real axes in the new coordinates defined by the map: the intuition results by replacing "real" by "complexified quaternionic"). The possibility to solve field equations in this manner would be of enormous significance since besides basic arithmetic operations also the functional decomposition of O_c-real-analytic functions produces similar functions. One could speak of the algebra of space-time surfaces.

What is remarkable that the complexified octonion real analytic functions are obtained by analytic continuation from single real valued function of real argument. The real functions form naturally a hierarchy of polynomials (maybe also rational functions) and number theoretic vision suggests that there coefficients are rationals or algebraic numbers. Already for rational coefficients hierarchy of algebraic extensions of rationals results as one solves the vanishing conditions. There is a temptation to regard this hierarchy coding for space-time sheets as an analog of DNA.

Note that in the recent formulation there is no need to pose separately the condition about integrable distribution of $M^2(x) \subset M^4$.

2.4.2 Quaternionicity condition for space-time surfaces

Quaternionicity actually has a surprisingly simple formulation at the level of space-time surfaces. The following discussion applies to both M^8 and H with minor modifications if one accepts that also H can allow octonionic tangent space structure, which does not require gamma matrices.

1. Quaternionicity is equivalent with associativity guaranteed by the vanishing of the associator $A(a,b,c) = a(bc) - (ab)c$ for any triplet of imaginary tangent vectors in the tangent space of the space-time surface. The condition must hold true for purely imaginary combinations of tangent vectors.

2. If one is able to choose the coordinates in such a way that one of the tangent vectors corresponds to real unit (in the embedding map embedding space M^4 coordinate depends only on the time coordinate of space-time surface), the condition reduces to the vanishing of the octonionic product of remaining three induced gamma matrices interpreted as octonionic gamma matrices. This condition looks very simple - perhaps too simple!- since it involves only first derivatives of the embedding space vectors.

   One can of course whether quaternionicity conditions replace field equations or only select preferred extremals. In the latter case, one should be able to prove that quaternionicity conditions are consistent with the field equations.

3. Field equations would reduce to tri-linear equations in in the gradients of embedding space coordinates (rather than involving embedding space coordinates quadratically). Sum of analogs of 3×3 determinants deriving from $a \times (b \times b)$ for different octonion units is involved.

4. Written explicitly field equations give in terms of vielbein projections e^A_α, vielbein vectors e^A_k, coordinate gradients $\partial_\alpha h^k$ and octonionic structure constants f_{ABC} the following conditions stating that the projections of the octonionic associator tensor to the space-time surface vanishes:

$$
\begin{aligned}
e^A_\alpha e^B_\beta e^C_\gamma A^E_{ABC} &= 0 \ , \\
A^E_{ABC} &= f_{AD}{}^E f_{BC}{}^D - f_{AB}{}^D f_{DC}{}^E \ , \\
e^A_\alpha &= \partial_\alpha h^k e^A_k \ , \\
\Gamma_k &= e^A_k \gamma_A \ .
\end{aligned}
\tag{2.1}
$$

   The very naïve idea would be that the field equations are indeed integrable in the sense that they reduce to these tri-linear equations. Tri-linearity in derivatives is highly non-trivial outcome simplifying the situation further. These equations can be formulated as the as purely algebraic equations written above plus integrability conditions

$$
F^A_{\alpha\beta} = D_\alpha e^A_\beta - D_\beta e^A_\alpha = 0 \ . \tag{2.2}
$$

   One could say that vielbein projections define an analog of a trivial gauge potential. Note however that the covariant derivative is defined by spinor connection rather than this effective gauge potential which reduces to that in SU(2). Similar formulation holds true for field equations and one should be able to see whether the field equations formulated in terms of derivatives of vielbein projections commute with the associatitivity conditions.

5. The quaternionicity conditions can be formulated as vanishing of generalization of Cayley's hyperdeterminant for "hypermatrix" a_{ijk} with 2-valued indiced (see `http://tinyurl.com/ya7h3n9z`). Now one has 8 hyper-matrices with 3 8-valued indices associated with the vanishing $A^E_{BCD}x^By^Cz^D = 0$ of trilinear forms defined by the associators. The conditions say somethig only about the octonioni structure constants and since octonionic space allow quaternionic sub-spaces these conditions must be satisfied.

The inspection of the Fano triangle [A11] (see **Fig. 1**) expressing the multiplication table for octonionic imaginary units reveals that give any two imaginary octonion units e_1 and e_2 their product e_1e_2 (or equivalently commutator) is imaginary octonion unit (2 times octonion unit) and the three units span together with real unit quaternionic sub-algebra. There it seems that one can generate local quaternionic sub-space from two imaginary units plus real unit. This generalizes to the vielbein components of tangent vectors of space-time surface and one can build the solutions to the quaternionicity conditions from vielbein projections e_1, e_2, their product $e_3 = k(x)e_1e_2$ and real fourth "time-like" vielbein component which must be expressible as a combination of real unit and imaginary units:

$$e_0 = a \times 1 + b^i e_i$$

For static solutions this condition is trivial. Here summation over i is understood in the latter term. Besides these conditions one has integrability conditions and field equations for Kähler action. This formulation suggests that quaternionicity is additional - perhaps defining - property of preferred extremals.

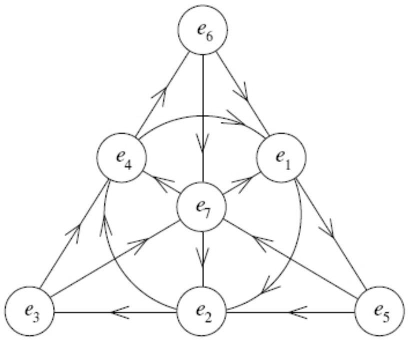

Figure 1: Octonionic triangle: the six lines and one circle containing three vertices define the seven associative triplets for which the multiplication rules of the ordinary quaternion imaginary units hold true. The arrow defines the orientation for each associative triplet. Note that the product for the units of each associative triplets equals to real unit apart from sign factor.

2.5 Quaternionicity At The Level Of Embedding Space Quantum Numbers

From the multiplication table of octonions as illustrated by Fano triangle [A11] one finds that all edges of the triangle, the middle circle and the three the lines connecting vertices to the midpoints of opposite side define triplets of quaternionic units. This means that by taking real unit and any imaginary unit in quaternionic M^4 algebra spanning $M^2 \subset M^4$ and two imaginary units in the complement representing CP_2 tangent space one obtains quaternionic algebra. This suggests an explanation for the preferred M^2 contained in tangent space of space-time surface (the M^2: s could form an integrable distribution). Four-momentum restricted to M^2 and I_3 and Y interpreted as tangent vectors in CP_2 tangent space defined quaterionic sub-algebra. This could give content for the idea that quantum numbers are quaternionic.

I have indeed proposed that the four-momentum belongs to M^2. If $M^2(x)$ form a distribution as the proposal for the preferred extremals suggests this could reflect momentum exchanges between different points of the space-time surface such that total momentum is conserved or momentum exchange between two sheets connected by wormhole contacts.

2.6 Questions

In following some questions related to $M^8 - H$ duality are represented.

2.6.1 Could associativity condition be formulated using modified gamma matrices?

Skeptic can criticize the minimal form of $M^8 - H$ duality involving no Kähler action in M^8 is unrealistic. Why just Kähler action? What makes it so special? The only defense that I can imagine is that Kähler action is in many respects unique choice.

An alternative approach would replace induced gamma matrices with the modified ones to get the correlation In the case of M^8 this option cannot work. One cannot exclude it for H.

1. For Kähler action the Kähler-Dirac gamma matrices $\Gamma^\alpha = \frac{\partial L_K}{\partial h^k_\alpha}\Gamma^k$, $\Gamma_k = e^A_k \gamma_A$, assign to a given point of X^4 a 4-D space which need not be tangent space anymore or even its sub-space.

   The reason is that canonical momentum current contains besides the gravitational contribution coming from the induced metric also the "Maxwell contribution" from the induced Kähler form not parallel to space-time surface. In the case of M^8 the duality map to H is therefore lost.

2. The space spanned by the Kähler-Dirac gamma matrices need not be 4-dimensional. For vacuum extremals with at most 2-D CP_2 projection Kähler-Dirac gamma matrices vanish identically. For massless extremals they span 1- D light-like subspace. For CP_2 vacuum extremals the modified gamma matrices reduces to ordinary gamma matrices for CP_2 and the situation reduces to the quaternionicity of CP_2. Also for string like objects the conditions are satisfied since the gamma matrices define associative sub-space as tangent space of $M^2 \times S^2 \subset M^4 \times CP_2$. It seems that associativity is satisfied by all known extremals. Hence Kähler-Dirac gamma matrices are flexible enough to realize associativity in H.

3. Kähler-Dirac gamma matrices in Dirac equation are required by super conformal symmetry for the extremals of action and they also guarantee that vacuum extremals defined by surfaces in $M^4 \times Y^2$, Y^2 a Lagrange sub-manifold of CP_2, are trivially hyper-quaternionic surfaces. The modified definition of associativity in H does not affect in any manner $M^8 - H$ duality necessarily based on induced gamma matrices in M^8 allowing purely number theoretic interpretation of standard model symmetries. One can however argue that the most natural definition of associativity is in terms of induced gamma matrices in both M^8 and H.

Remark: A side comment not strictly related to associativity is in order. The anti-commutators of the Kähler-Dirac gamma matrices define an effective Riemann metric and one can assign to it the counterparts of Riemann connection, curvature tensor, geodesic line, volume, etc... One would have two different metrics associated with the space-time surface. Only if the action defining space-time surface is identified as the volume in the ordinary metric, these metrics are equivalent. The index raising for the effective metric could be defined also by the induced metric and it is not clear whether one can define Riemann connection also in this case. Could this effective metric have concrete physical significance and play a deeper role in quantum TGD? For instance, AdS-CFT duality leads to ask whether interactions be coded in terms of the gravitation associated with the effective metric.

Now skeptic can ask why should one demand $M^8 - H$ correspondence if one in any case is forced to introduced Kähler also at the level of M^8? Does $M^8 - H$ correspondence help to construct preferred extremals or does it only bring in a long list of conjectures? I can repeat the questions of the skeptic.

2.6.2 Minkowskian-Euclidian $\leftrightarrow$ associative–co-associative?

The 8-dimensionality of M^8 allows to consider both associativity of the tangent space and associativity of the normal space- let us call this co-associativity of tangent space- as alternative options. Both options are needed as has been already found. Since space-time surface decomposes into regions whose induced metric possesses either Minkowskian or Euclidian signature, there is a strong temptation to propose that Minkowskian regions correspond to associative and Euclidian regions to co-associative regions so that space-time itself would provide both the description and its dual.

The proposed interpretation of conjectured associative-co-associative duality relates in an interesting manner to p-adic length scale hypothesis selecting the primes $p \simeq 2^k$, k positive integer as preferred p-adic length scales. $L_p \propto \sqrt{p}$ corresponds to the p-adic length scale defining the size of the space-time sheet at which elementary particle represented as CP_2 type extremal is topologically condensed and is of order Compton length. $L_k \propto \sqrt{k}$ represents the p-adic length scale of the wormhole contacts associated with the CP_2 type extremal and CP_2 size is the natural length unit now. Obviously the quantitative formulation for associative-co-associative duality would be in terms $p \to k$ duality.

2.6.3 Can $M^8 - H$ duality be useful?

Skeptic could of course argue that $M^8 - H$ duality generates only an inflation of unproven conjectures. This might be the case. In the following I will however try to defend the conjecture. One can however find good motivations for $M^8 - H$ duality: both theoretical and physical.

1. If $M^8 - H$ duality makes sense for induced gamma matrices also in H, one obtains infinite sequence if dualities allowing to construct preferred extremals iteratively. This might relate to octonionic real-analyticity and composition of octonion-real-analytic functions.

2. $M^8 - H$ duality could provide much simpler description of preferred extremals of Kähler action as hyper-quaternionic surfaces. Unfortunately, it is not clear whether one should introduce the counterpart of Kähler action in M^8 and the coupling of M^8 spinors to Kähler form. Note that the Kähler form in E^4 would be self dual and have constant components: essentially parallel electric and magnetic field of same constant magnitude.

3. $M^8 - H$ duality provides insights to low energy physics, in particular low energy hadron physics. M^8 description might work when H-description fails. For instance, perturbative QCD which corresponds to H-description fails at low energies whereas M^8 description might become perturbative description at this limit. Strong $SO(4) = SU(2)_L \times SU(2)_R$ invariance is the basic symmetry of the phenomenological low energy hadron models based on conserved vector current hypothesis (CVC) and partially conserved axial current hypothesis (PCAC). Strong $SO(4) = SU(2)_L \times SU(2)_R$ relates closely also to electro-weak gauge group $SU(2)_L \times U(1)$ and this connection is not well understood in QCD description. $M^8 - H$ duality could provide this connection. Strong $SO(4)$ symmetry would emerge as a low energy dual of the color symmetry. Orbital $SO(4)$ would correspond to strong $SU(2)_L \times SU(2)_R$ and by flatness of E^4 spin like $SO(4)$ would correspond to electro-weak group $SU(2)_L \times U(1)_R \subset SO(4)$. Note that the inclusion of coupling to Kähler gauge potential is necessary to achieve respectable spinor structure in CP_2. One could say that the orbital angular momentum in $SO(4)$ corresponds to strong isospin and spin part of angular momentum to the weak isospin.

   This argument does not seem to be consistent with $SU(3) \times U(1) \subset SU(4)$ symmetry for Mx Dirac equation. One can however argue that $SU(4)$ symmetry combines $SO(4)$ multiplets together. Furthermore, $SO(4)$ represents the isometries leaving Kähler form invariant.

2.6.4 $M^8 - H$ duality in low energy physics and low energy hadron physics

$M^8 - H$ can be applied to gain a view about color confinement. The basic idea would be that $SO(4)$ and $SU(3)$ provide provide dual descriptions of quarks using E^4 and CP_2 partial waves and low energy hadron physics corresponds to a situation in which M^8 picture provides the perturbative approach whereas H picture works at high energies.

A possible interpretation is that the space-time surfaces vary so slowly in CP_2 degrees of freedom that can approximate CP_2 with a small region of its tangent space E^4. One could also say that color interactions mask completely electroweak interactions so that the spinor connection of CP_2 can be neglected and one has effectively E^4. The basic prediction is that $SO(4)$ should appear as dynamical symmetry group of low energy hadron physics and this is indeed the case.

Consider color confinement at the long length scale limit in terms of $M^8 - H$ duality.

1. At high energy limit only lowest color triplet color partial waves for quarks dominate so that QCD description becomes appropriate whereas very higher color partial waves for quarks and gluons are expected to appear at the confinement limit. Since WCW degrees of freedom begin to dominate, color confinement limit transcends the descriptive power of QCD.

2. The success of $SO(4)$ sigma model in the description of low lying hadrons would directly relate to the fact that this group labels also the E^4 Hamiltonians in M^8 picture. Strong $SO(4)$ quantum numbers can be identified as orbital counterparts of right and left handed electro-weak isospin coinciding with strong isospin for lowest quarks. In sigma model pion and sigma boson form the components of E^4 valued vector field or equivalently collection of four E^4 Hamiltonians corresponding to spherical E^4 coordinates. Pion corresponds to S^3 valued unit vector field with charge states of pion identifiable as three Hamiltonians defined by the coordinate components. Sigma is mapped to the Hamiltonian defined by the E^4 radial coordinate. Excited mesons corresponding to more complex Hamiltonians are predicted.

3. The generalization of sigma model would assign to quarks E^4 partial waves belonging to the representations of $SO(4)$. The model would involve also 6 $SO(4)$ gluons and their $SO(4)$ partial waves. At the low energy limit only lowest representations would be important whereas at higher energies higher partial waves would be excited and the description based on CP_2 partial waves would become more appropriate.

4. The low energy quark model would rely on quarks moving $SO(4)$ color partial waves. Left *resp.* right handed quarks could correspond to $SU(2)_L$ *resp.* $SU(2)_R$ triplets so that spin statistics problem would be solved in the same manner as in the standard quark model.

5. Family replication phenomenon is described in TGD framework the same manner in both cases so that quantum numbers like strangeness and charm are not fundamental. Indeed, p-adic mass calculations allowing fractally scaled up versions of various quarks allow to replace Gell-Mann mass formula with highly successful predictions for hadron masses [K26].

To my opinion these observations are intriguing enough to motivate a concrete attempt to construct low energy hadron physics in terms of $SO(4)$ gauge theory.

2.7 Summary

The overall conclusion is that the most convincing scenario relies on the associativity/co-associativity of space-time surfaces define by induced gamma matrices and applying both for M^8 and H. The fact that the duality can be continued to an iterated sequence of duality maps $M^8 \rightarrow H \rightarrow H...$ is what makes the proposal so fascinating and suggests connection with fractality.

The introduction of Kähler action and coupling of spinors to Kähler gauge potentials is highly natural. One can also consider the idea that the space-time surfaces in M^8 and H have same induced metric and Kähler form: for iterated duality map this would mean that the steps in the map produce space-time surfaces which identical metric and Kähler form so that the sequence might stop. M^8_H duality might provide two descriptions of same underlying dynamics: M^8 description would apply in long length scales and H description in short length scales.

3 Quaternions and TGD

3.1 Are Euclidian Regions Of Preferred Extremals Quaternion- Kähler Manifolds?

In blog comments Anonymous gave a link to an article (see `http://tinyurl.com/y7j9hxr8`) about construction of 4-D quaternion-Kähler metrics with an isometry: they are determined by so called

$SU(\infty)$ Toda equation. I tried to see whether quaternion-Kähler manifolds could be relevant for TGD.

From Wikipedia (see http://tinyurl.com/yd8feoev) one can learn that QK is characterized by its holonomy, which is a subgroup of $Sp(n) \times Sp(1)$: $Sp(n)$ acts as linear symplectic transformations of $2n$-dimensional space (now real). In 4-D case tangent space contains 3-D sub-manifold identifiable as imaginary quaternions. CP_2 is one example of QK manifold for which the subgroup in question is $SU(2) \times U(1)$ and which has non-vanishing constant curvature: the components of Weyl tensor represent the quaternionic imaginary units. QKs are Einstein manifolds: Einstein tensor is proportional to metric.

What is really interesting from TGD point of view is that twistorial considerations show that one can assign to QK a special kind of twistor space (twistor space in the mildest sense requires only orientability). Wiki tells that if Ricci curvature is positive, this (6-D) twistor space is what is known as projective Fano manifold with a holomorphic contact structure. Fano variety has the nice property that as (complex) line bundle (the twistor space property) it has enough sections to define the embedding of its base space to a projective variety. Fano variety is also complete: this is algebraic geometric analogy of topological property known as compactness.

3.1.1 QK manifolds and twistorial formulation of TGD

How the QKs could relate to the twistorial formulation of TGD?

1. In the twistor formulation of TGD [K39] the space-time surfaces are 4-D base spaces of 6-D twistor spaces in the Cartesian product of 6-D twistor spaces of M^4 and CP_2 - the only twistor spaces with Kähler structure. In TGD framework space-time regions can have either Euclidian or Minkowskian signature of induced metric. The lines of generalized Feynman diagrams are Euclidian.

2. Could the twistor spaces associated with the lines of generalized Feynman diagrams be projective Fano manifolds? Could QK structure characterize Euclidian regions of preferred extremals of Kähler action? Could a generalization to Minkowskian regions exist.

   I have proposed that so called Hamilton-Jacobi structure [K45] characterizes preferred extremals in Minkowskian regions. It could be the natural Minkowskian counterpart for the quaternion Kähler structure, which involves only imaginary quaternions and could make sense also in Minkowski signature. Note that unit sphere of imaginary quaternions defines the sphere serving as fiber of the twistor bundle.

Why it would be natural to have QK that is corresponding twistor space, which is projective contact Fano manifold?

1. QK property looks very strong condition but might be true for the preferred extremals satisfying very strong conditions stating that the classical conformal charges associated with various conformal algebras extending the conformal algebras of string models [K45], [L7]. These conditions would be essentially classical gauge conditions stating that strong form of holography implies by strong form of General Coordinate Invariance (GCI) is realized: that is partonic 2-surfaces and their 4-D tangent space data code for quantum physics.

2. Kähler property makes sense for space-time regions of Euclidian signature and would be natural is these regions can be regarded as small deformations of CP_2 type vacuum extremals with light-like M^4 projection and having the same metric and Kähler form as CP_2 itself.

3. Fano property implies that the 4-D Euclidian space-time region representing line of the Feynman diagram can be imbedded as a sub-manifold to complex projective space CP_n. This would allow to use the powerful machinery of projective geometry in TGD framework. This could also be a space-time correlate for the fact that CP_ns emerge in twistor Grassmann approach expected to generalize to TGD framework.

4. CP_2 allows both projective (trivially) and contact (even symplectic) structures. $\delta M^4_+ \times CP_2$ allows contact structure - I call it loosely symplectic structure. Also 3-D light-like orbits of partonic 2-surfaces allow contact structure. Therefore holomorphic contact structure for the twistor space is natural.

5. Both the holomorphic contact structure and projectivity of CP_2 would be inherited if QK property is true. Contact structures at orbits of partonic 2-surfaces would extend to holomorphic contact structures in the Euclidian regions of space-time surface representing lines of generalized Feynman diagrams. Projectivity of Fano space would be also inherited from CP_2 or its twistor space $SU(3)/U(1)\times U(1)$ (flag manifold identifiable as the space of choices for quantization axes of color isospin and hypercharge).

The article considers a situation in which the QK manifold allows an isometry. Could the isometry (or possibly isometries) for QK be seen as a remnant of color symmetry or rotational symmetries of M^4 factor of embedding space? The only remnant of color symmetry at the level of embedding space spinors is anomalous color hyper charge (color is like orbital angular momentum and associated with spinor harmonic in CP_2 center of mass degrees of freedom). Could the isometry correspond to anomalous hypercharge?

3.1.2 How to choose the quaternionic imaginary units for the space-time surface?

Parallellizability is a very special property of 3-manifolds allowing to choose quaternionic imaginary units: global choice of one of them gives rise to twistor structure.

1. The selection of time coordinate defines a slicing of space-time surface by 3-surfaces. GCI would suggest that a generic slicing gives rise to 3 quaternionic units at each point each 3-surface? The parallelizability of 3-manifolds - a unique property of 3-manifolds - means the possibility to select global coordinate frame as section of the frame bundle: one has 3 sections of tangent bundle whose inner products give rose to the components of the metric (now induced metric) guarantees this. The tri-bein or its dual defined by two-forms obtained by contracting tri-bein vectors with permutation tensor gives the quanternionic imaginary units. The construction depends on 3-metric only and could be carried out also in GRT context. Note however that topology change for 3-manifold might cause some non-trivialities. The metric 2-dimensionality at the light-like orbits of partonic 2-surfaces should not be a problem for a slicing by space-like 3-surfaces. The construction makes sense also for the regions of Minkowskian signature.

2. In fact, any 4-manifold (see `http://tinyurl.com/yb8l34b5`) [A13] allows almost quaternionic as the above slicing argument relying on parallelizibility of 3-manifolds strongly suggests.

3. In zero energy ntology (ZEO)- a purely TGD based feature - there are very natural special slicings. The first one is by linear time-like Minkowski coordinate defined by the direction of the line connecting the tips of the causal diamond (CD). Second one is defined by the light-cone proper time associated with either light-cone in the intersection of future and past directed light-cones defining CD. Neither slicing is global as it is easy to see.

3.1.3 The relationship to quaternionicity conjecture and M^8-H duality

One of the basic conjectures of TGD is that preferred extremals consist of quaternionic/ co-quaternionic (associative/co-associative) regions [K37]. Second closely related conjecture is M^8-H duality allowing to map quaternionic/co-quaternionic surfaces of M^8 to those of $M^4\times CP_2$. Are these conjectures consistent with QK in Euclidian regions and Hamilton-Jacobi property in Minkowskian regions? Consider first the definition of quaternionic and co-quaternionic space-time regions.

1. Quaternionic/associative space-time region (with Minkowskian signature) is defined in terms of induced octonion structure obtained by projecting octonion units defined by vielbein of $H=M^4\times CP_2$ to space-time surface and demanding that the 4 projections generate quaternionic sub-algebra at each point of space-time.

   If there is also unique complex sub-algebra associated with each point of space-time, one obtains one can assign to the tangent space-of space-time surface a point of CP_2. This allows to realize M^8-H duality [K37] as the number theoretic analog of spontaneous compactification (but involving no compactification) by assigning to a point of $M^4 = M^4 \times CP_2$ a point of $M^4 \times CP_2$. If the image surface is also quaternionic, this assignment makes sense also for space-time surfaces in H so that $M^8 - H$ duality generalizes to $H - H$ duality allowing to assign to given preferred extremal a hierarchy of extremals by iterating this assignment. One obtains a category with morphisms identifiable as these duality maps.

2. Co-quaternionic/co-associative structure is conjectured for space-time regions of Euclidian signature and 4-D CP_2 projection. In this case normal space of space-time surface is quaternionic/associative. A multiplication of the basis by preferred unit of basis gives rise to a quaternionic tangent space basis so that one can speak of quaternionic structure also in this case.

3. Quaternionicity in this sense requires unique identification of a preferred time coordinate as embedding space coordinate and corresponding slicing by 3-surfaces and is possible only in TGD context. The preferred time direction would correspond to real quaternionic unit. Preferred time coordinate implies that quaternionic structure in TGD sense is more specific than the QK structure in Euclidian regions.

4. The basis of induced octonionic imaginary unit allows to identify quaternionic imaginary units linearly related to the corresponding units defined by tri-bein vectors. Note that the multiplication of octonionic units is replaced with multiplication of antisymmetric tensors representing them when one assigns to the quaternionic structure potential QK structure. Quaternionic structure does not require Kähler structure and makes sense for both signatures of the induced metric. Hence a consistency with QK and its possible analog in Minkowskian regions is possible.

5. The selection of the preferred imaginary quaternion unit is necessary for $M^8 - H$ correspondence. This selection would also define the twistor structure. For quaternion-Kähler manifold this unit would be covariantly constant and define Kähler form - maybe as the induced Kähler form.

6. Also in Minkowskian regions twistor structure requires a selection of a preferred imaginary quaternion unit. Could the induced Kähler form define the preferred imaginary unit also now? Is the Hamilton-Jacobi structure consistent with this?

   Hamilton-Jacobi structure involves a selection of 2-D complex plane at each point of space-time surface. Could induced Kähler magnetic form for each 3-slice define this plane? It is not necessary to require that 3-D Kähler form is covariantly constant for Minkowskian regions. Indeed, massless extremals representing analogs of photons are characterized by local polarization and momentum direction and carry time-dependent Kähler-electric and -magnetic fields. One can however ask whether monopole flux tubes carry covariantly constant Kähler magnetic field: they are indeed deformations of what I call cosmic strings [K6, K11] for which this condition holds true?

3.2 The Notion of Quaternion Analyticity

The 4-D generalization of conformal invariance suggests strongly that the notion of analytic function generalizes somehow. The obvious ideas coming in mind are appropriately defined quaternionic and octonion analyticity. I have used a considerable amount of time to consider these possibilities but had to give up the idea about octonion analyticity could somehow allow to preferred extemals.

3.2.1 Basic idea

One can argue that quaternion analyticity is the more natural option in the sense that the local octonionic embedding space coordinate (or at least M^8 or E^8 coordinate, which is enough if $M^8 - H$ duality holds true) would for preferred extremals be expressible in the form

$$o(q) = u(q) + v(q) \times I \quad . \tag{3.1}$$

Here q is quaternion serving as a coordinate of a quaternionic sub-space of octonions, and I is octonion unit belonging to the complement of the quaternionic sub-space, and multiplies $v(q)$ from *right* so that quaternions and quaternionic differential operators acting from left do not notice these coefficients at all. A stronger condition would be that the coefficients are real. $u(q)$ and $v(q)$ would be quaternionic Taylor- of even Laurent series with coefficients multiplying powers of q from right for the same reason.

The signature of M^4 metric is a problem. I have proposed a complexification of M^8 and M^4 to get rid of the problem by assuming that the embedding space corresponds to surfaces in the space M^8 identified as octonions of form $o_8 = Re(o) + iIm(o)$, where o is imaginary part of ordinary octonion and i is commuting imaginary unit. M^4 would correspond to quaternions of form $q_4 = Re(q) + iIm(q)$. What is important is that powers of q_4 and o_8 belong to this sub-space (as follows from the vanishing of cross product term in the square of octonion/quaternion) so that powers of q_4 (o_8) has imaginary part proportional to $Im(q)$ ($Im(o)$)

I ended up to reconsider the idea of quaternion analyticity after having found two very interesting articles discussing the generalization of Cauchy-Riemann equations. The first article (see `http://tinyurl.com/yb8l34b5`) [A13] was about so called triholomorphic maps between 4-D almost quaternionic manifolds. The article gave as a reference an article (see `http://tinyurl.com/y7kww2o2`) [A12] about quaternionic analogs of Cauchy-Riemann conditions discussed by Fueter long ago (somehow I have managed to miss Fueter's work just like I missed Hitchin's work about twistorial uniqueness of M^4 and CP_2), and also a new linear variant of these conditions, which seems especially interesting from TGD point of view as will be found.

3.2.2 The first form of Cauchy-Riemann-Fueter conditions

Cauchy-Riemann-Fueter (CRF) conditions generalize Cauchy-Riemann conditions. These conditions are however not unique. Consider first the translationally invariant form of CRF conditions.

1. The translationally invariant form of CRF conditions is $\partial_{\overline{q}} f = 0$ or explicitly

$$\partial_{\overline{q}} f = d_1 f + d_2 f \equiv (\partial_t - \partial_x I) f - (\partial_y J + \partial_z K) f = 0 \ . \tag{3.2}$$

   This form is not unique: one can perform SO(3) rotations of the quaternionic imaginary units acting as automorphisms of quaternions. This form does not allow quaternionic Taylor series as a solution. Note that the Taylor coefficients multiplying powers of the coordinate from right are arbitrary quaternions. What looks pathological is that even linear functions of q fail be solve this condition. What is however interesting that in flat space the equation is equivalent with Dirac equation for a pair of Majorana spinors [A13].

   Function $f = t + Ix - Jy - Kz$ is perhaps the simplest solution to the condition. One can define also other variants of $\overline{q}$, in particular the variant $\overline{q} = t + Ix - Jy - Kz$ giving $f = t + Ix + Jy + Kz$ as a solution.

2. The condition allows functions depending on complex coordinate z of some complex-plane only. It also allows functions satisfying two separate analyticity conditions, say $d_1 f = 0$ and $d_2 f = 0$, say

$$\partial_{\overline{u}} f = (\partial_t - \partial_x I) f = 0 \ ,$$
$$\partial_{\overline{v}} f = -(\partial_y J + \partial_z K) f = -J(\partial_y - \partial_z I) f = 0 \ . \tag{3.3}$$

   In the latter formula J multiplies from *left*! One has good hopes of obtaining holomorphic functions of two complex coordinates.

The simplest solution to the conditions is complex value function $f(u = x+iy, v = y+iz)$ of two complex variables. The image of E^4 is 2-dimensional whereas for $f_0 = t + Ix - Jy - Kz$ it is 4-D.

In Euclidian signature one obtains quaternion valued map if the Taylor coefficients a_{mn} in the series of $f(u,v)$ are quaternions and are taken to the right: $q = f(u,v) = \sum u^m v^n a_{mn}$ to avoid problems from non-commutativity. With this assumption the image would be 4-D in the generic case.

In TGD one must consider Minkowskian signature and it turns out that the situation changes dramatically, and the naïve view about quaternion analyticity must be given up. The experience about extermals of Kähler action suggests a modification of the analyticity properties consistent with the signature but whether one should call this analyticity quaternion analyticity is a matter of taste.

3.2.3 Second form of CRF conditions

Second form of CRF conditions proposed in [A12] is tailored in order to realize the almost obvious manner to realize quaternion analyticity.

1. The ingenious idea is to replace preferred quaternionic imaginary unit by a imaginary unit which is in radial direction: $e_r = (xI + yJ + zK)/r$ and require analyticity with respect to the coordinate $t + e_r r$. The solution to the condition is power series in $t + e_r r = q$ so that one obtains quaternion analyticity.

2. The explicit form of the conditions is

$$(\partial_t - e_r \partial_r) f = (\partial_t - \tfrac{e_r}{r} r\partial_r) f = 0 \quad . \tag{3.4}$$

   This form allows both the desired quaternionic Taylor series and ordinary holomorphic functions of complex variable in one of the 3 complex coordinate planes as general solutions.

3. This form of CRF is neither Lorentz invariant nor translationally invariant but remains invariant under simultaneous scalings of t and r and under time translations. Under rotations of either coordinates or of imaginary units the spatial part transforms like vector so that quaternionic automorphism group $SO(3)$ serves as a moduli space for these operators.

4. The interpretation of the latter solutions inspired by ZEO would be that in Minkowskian regions r corresponds to the light-like radial coordinate of the either boundary of CD, which is part of $\delta M^4_\pm$. The radial scaling operator is that assigned with the light-like radial coordinate of the light-cone boundary. A slicing of CD by surfaces parallel to the $\delta M^4_\pm$ is assumed and implies that the line $r = 0$ connecting the tips of CD is in a special role. The line connecting the tips of CD defines coordinate line of time coordinate. The breaking of rotational invariance corresponds to the selection of a preferred quaternion unit defining the twistor structure and preferred complex sub-space.

   In regions of Euclidian signature r could correspond to the radial Eguchi-Hanson coordinate of CP_2 and $r = 0$ corresponds to a fixed point of $U(2)$ subgroup under which CP_2 complex coordinates transform linearly.

5. Also in this case one can ask whether solutions depending on two complex local coordinates analogous to those for translationally invariant CRF condition are possible. The remain imaginary units would be associated with the surface of sphere allowing complex structure.

3.2.4 Generalization of CRF conditions?

Could the proposed forms of CRF conditions be special cases of much more general CRF conditions as CR conditions are?

1. Ordinary complex analysis suggests that there is an infinite number of choices of the quaternionic coordinates related by the above described quaternion-analytic maps with 4-D images. The form of of the CRF conditions would be different in each of these coordinate systems and would be obtained in a straightforward manner by chain rule.

2. One expects the existence of large number of different quaternion-conformal structures not related by quaternion-analytic transformations analogous to those allowed by higher genus Riemann surfaces and that these conformal equivalence classes of four-manifolds are characterized by a moduli space and the analogs of Teichmüller parameters depending on 3-topology. In TGD framework strong form of holography suggests that these conformal equivalence classes for preferred extremals could reduce to ordinary conformal classes for the partonic 2-surfaces. An attractive possibility is that by conformal gauge symmetries the functional integral over WCW reduces to the integral over the conformal equivalence classes.

3. The quaternion-conformal structures could be characterized by a standard choice of quaternionic coordinates reducing to the choice of a pair of complex coordinates. In these coordinates the general solution to quaternion-analyticity conditions would be of form described for the linear ansatz. The moduli space corresponds to that for complex or hyper-complex structures defined in the space-time region.

3.2.5 Geometric formulation of the CRF conditions

The previous naïve generalization of CRF conditions treats imaginary units without trying to understand their geometric content. This leads to difficulties when when tries to formulate these conditions for maps between quaternionic and hyper-quaternionic spaces using purely algebraic representation of imaginary units since it is not clear how these units relate to each other.

In [A13] the CRF conditions are formulated in terms of the antisymmetric (1, 1) type tensors representing the imaginary units: they exist for almost quaternionic structure. One might hope that this so also for the almost hyper-quaternionic structure needed in Minkowskian signature.

The generalization of CRF conditions is proposed in terms of the Jacobian J of the map mapping tangent space TM to TN and antisymmetric tensors J_u and j_u representing the quaternionic imaginary units of N and M respectively. The generalization of CRF conditions reads as

$$J - \sum_u J_u \circ J \circ j_u = 0 \quad . \tag{3.5}$$

For $N = M$ it reduces to the translationally invariant algebraic form of the conditions discussed above. These conditions reduce to CR conditions in 2-D case when one has only single J_u. In quaternionic case this form is only replaced with sum over all 3 antisymmetric forms representing quaternionic units.

These conditions are not unique. One can perform an SO(3) rotation (quaternion automorphism) of the imaginary units mediated by matrix Λ^{uv} to obtain

$$J - \Lambda^{uv} J_u \circ J \circ j_v = 0 \quad . \tag{3.6}$$

The matrix Λ can depend on point so that one has a kind of gauge symmetry. The most general triholomorphic map allows the presence of Λ. Note that these conditions make sense on any coordinates and complex analytic maps generate new forms of these conditions.

In Minkowskian signature one would have 3 forms iJ_u serving as counterparts for iI, iJ, iK. The most natural possibility is that i is represented as algebraic unit and I, J, K as antisymmetry self-dual em fields with E and B constant and parallel to each other and normalize to have unit lengths. Their directions would correspond to 3 orthogonal coordinate axis. The twistor lift forces to introduce the generalization of Kähler form of M^4 and one can introduce all these 3 independent forms as counterpart of hyperquaternionic units: they are introduced also for ordinary twistor structure but one of them is selected as a preferred one. The only change in the conditions is change of sign of the sign of the sum coming from $i^2 = -1$ so that one has

$$J + \sum_u J_u \circ J \circ j_u = 0 \ . \tag{3.7}$$

These conditions are therefore formally well-defined also when one maps quaternionic to hyper-quaternionic space or vice versa.

In 2-dimensional hypercomplex case the conditions allow to write hypercomplex map $X - Y = U = f(x - y)$ and $X + Y = V = f(x + y)$. In special case this solutions of massless d'Alembertian in M^2. Alternatively, one can express f as analytic function of $x + iIy$ and pick up $X - Y$ and $X + Y$. It is however not clear whether one can write a Taylor expansion in hyper-quaternionic coordinate in the similar manner.

Covariant forms of structure constant tensors reduce to octonionic structure constants and this allows to write the conditions explicitly. The index raising of the second index of the structure constants is however needed using the metrics of M and N. This complicates the situation and spoils linearity: in particular, for surfaces induced metric is needed. Whether local SO(3) rotation can eliminate the dependence on induced metric is an interesting question. Since M^4 imaginary units differ only by multiplication by i, Minkowskian structure constants differ only by sign from the Euclidian ones.

In the octonionic case the geometric generalization of CRF conditions does not seem to make sense. By non-associativity of octonion product it is not possible to have a matrix representation for the matrices so that a faithful representation of octonionic imaginary units as antisymmetric 1-1 forms does not make sense. If this representation exists it it must map octonionic associators to zero. Note however that CRF conditions do not involve products of three octonion units so that they make sense as algebraic conditions at least.

3.2.6 Does residue calculus generalize?

CRF conditions allow to generalize Cauchy formula allowing to express value of analytic function in terms of its boundary values [A13]. This would give a concrete realization of the holography in the sense that the physical variables in the interior could be expressed in terms of the data at the light-like partonic orbits and the ends of the space-time surface. Triholomorphic function satisfies d'Alembert/Laplace equations - in induced metric in TGD framework- so that the maximum modulus principle holds true. The general ansatz for a preferred extremals involving Hamilton-Jacobi structure leads to d'Alembert type equations for preferred extremals [K45].

Could the analog of residue calculus exist? Line integral would become 3-D integral reducing to a sum over poles and possible cuts inside the 3-D contour. The space-like 3-surfaces at the ends of CDs could define natural integration contours, and the freedom to choose contour rather freely would reflect General Coordinate Invariance. A possible choice for the integration contour would be the closed 3-surface defined by the union of space-like surfaces at the ends of CD and by the light-like partonic orbits.

Poles and cuts would be in the interior of the space-time surface. Poles have co-dimension 2 and cuts co-dimension 1. Strong form of holography suggests that partonic 2-surfaces and perhaps also string world sheets serve as candidates for poles. Light-like 3-surfaces (partonic orbits) defining the boundaries between Euclidian and Minkowskian regions are singular objects and could serve as cuts. The discontinuity would be due to the change of the signature of the induced metric. There are CDs inside CDs and one can also consider the possibility that sub-CDs define cuts, which in turn reduce to cuts associated with sub-CDs.

3.3 Are Preferred Extremals Quaternion-Analytic in Some Sense?

At what level quaternion analyticity could appear in TGD framework? Does it appear only in the formulation of conformal algebras and replace loop algebra with double loop algebra (roughly

$z^m \to u^m v^n$)? Or does it appear in some form also at the level of preferred extremals for which geometric form of quaternionicity is expected to appear - at least at the level of M^8?

3.3.1 Minimalistic view

Before continuing it is good to bring in mind the minimal assumptions and general vision.

1. If $M^8 - H$ duality [K37] holds true, the space-time surface $X^4 \subset M^8 = M^4 \times E^4$ is quaternionic surface in the sense that it have quaternionic tangent space and contains preferred $M^2 \subset M^4$ as part of their tangent space or more generally the 2-D hyper-complex subspaces $M^2(x)$ define and integrable distribution defining 2-D surface.

2. Quaternionicity in geometric sense in M^8 alone *implies* the interpretation as a 4-D surface in $H = M^4 \times CP_2$. There is *no need* to assume quaternionicity in geometric sense in H although it cannot be excluded and would have strong implications [K37]. This one should remember in order to avoid drowning to an inflation of speculations.

It is not at all clear what quaternion analyticity in Minkowskian signature really means or whether it is even possible. The skeptic inside me has a temptation to conclude that the direct extrapolation of quaternion analyticity from Euclidian to Minkowskian signature for space-time surfaces in H is not necessary and might be even impossible. On the other hand, the properties of the known extremals strongly suggest its analog. Quaternion analyticity could however appear at the tangent space level for various generalized conformal algebras transformed to double loop algebras for the proposed realization of the quaternion analyticity.

3.3.2 The naïve generalization of quaternion analyticity to Minkowski signature fails

Quaternion analyticity works nicely in Euclidian signature for maps $E^4 \to E^4$. One can also consider quaternion analytic maps $E^4 \to E^8$ with E^8 regarded as octonionic space of form $E^4 \oplus E^4 J$, where E^4 is quaternionic space and J is octonion unit in the complement of $E^4 \subset E^8$. The maps decompose to sums $f_1 \oplus f_2 J$ where f_i are quaternion analytic maps $E^4 \to E^4$. Consider maps $f : E^4 \to E^8$, whose graph should define Euclidian space-time surface.

1. One can construct octonion valued maps $f(u,v) = f_0 + \sum u^m v^n a_{mn} : E^4 \to E^8$ with E^4 identified as quaternionic sub-space of E^8. Recall that one has $u = t + Iz$, $v = (x + Iy)J$. a_{mn} can be octonions with the proposed definition of the Taylor series. Since each power $u^m v^n$ is analytic function, one still has quaternion analyticity in the proposed sense. The image would be 4-D in the general case.

2. By linearity the solutions obey linear superposition. They can be also multiplied if the product is defined as ordered product in such a way that only the powers $t+ix$ and $y+iz$ are multiplied together at left and coefficients a_{mn} are multiplied together at right. The analogy with quantum non-commutativity is obvious.

Can one generalize this ansatz to Minkowskian signature? One can try to look the ansatz for the embedding $X^4 \subset M^8 = M^4 \times E^4 J$ as sum $f = (f_1, f_2)$ of quaternion analytic maps $f_1 : X^4 \to M^4$ and $f_2 : X^4 \to E^4$. The general quaternion analytic ansatz for $X^4 \subset E^8$ fails due to the non-commutativity of quaternions.

The comparison of 2-dimensional hypercomplex case with 4-D hyperquaternionic case reveals the basic problem.

1. The analogs CR conditions allow to write hypercomplex map $X - Y = U = f(x - y)$ and $X + Y = V = f(x + y)$. In special case this gives the solutions of massless d'Alembertian in M^2 as sum of these solutions. Alternatively, one can express f as analytic function of $x + iIy$ and pick up $X - Y$ and $X + Y$. The use of hypercomplex numbers and hypercomplex analyticity is equivalent with use of functions $f(x - y)$ or $f(x + y)$.

2. The essential point is that for M^2 regarded as a sub-space of "complexified" complex numbers $z_1 + iz_2$ consisting of points $x + iIy$, the multiplication of numbers of form $x + iIy$ does not

lead out of M^2. For M^4 this is not anymore the case since $iI \times iJ = -K$ does not belong to the Minkowskian subspace of complexified quaternions. Hence there are no hopes about the existence of the analog of $f(z) = \sum a_n z^n$. For this reason also non-trivial powers $u^m v^n$ are excluded and one cannot build a Minkowskian generalization of quaternion analytic power series.

3. If one can allow the values of hyper-quaternion analytic functions to be in M^4_c rather than M^4, there are no problems but if one wants to represents space-time surfaces as graphs of hyper-quaternion analytic maps $f : M^4 \rightarrow M^8$ one must pose strong restrictions on allowed functions.

The restrictions on the allowed hyper-quaternion analytic functions look rather obvious for what might be called hyper-quaternion analytic maps $M^4 \rightarrow M^4$.

1. Assume a decomposition $M^4 = M^2 \times E^2$ such one has $f = (f_1, f_2)$, where $f_1 : M^2 \rightarrow M^2$ is analytic in hyper-complex sense and $f_2 : E^2 \rightarrow E^2$ is analytic in complex sense. Both these options are possible. One can write the map as $f(u, v) = f_1(u = t + iIz) + f_2(v = x + Iy)iJ$ and it satisfies the usual conditions $\partial_{\overline{u}} f = 0$ and $\partial_{\overline{v}} f = 0$. Note that iJ is taken to the right so that the differential operators acting from left in the analyticity conditions does not "notice" it.

   Linear superpositions of this kind of solutions with real coefficients are possible. One can multiply this kind of solutions if the multiplication is done separately in the Cartesian factors. Also functional composition is possible in the factors.

2. A generalization of the solution ansatz to integrable decompositions $M^4 = M^2(x) \oplus E^2(x)$ is rather plausible. This would mean a foliation of M^4 by pairs of 2-D surfaces. String world sheets and partonic 2-surfaces would be the physical counterpart for these foliations. I have called this kind of foliation Hamilton-Jacobi structure [K3] and it would serve as a generalization of the complex structure to 4-D Minkowskian case. In Euclidian signature it corresponds to ordinary complex structure in 4-D sense.

3. The analogy of double loop Lie algebra replacing powers z^m with $u^m v^n$ does does not however seem to be possible. Could this relate to SH forcing to code data using only functions of u or v and to select either string world sheet or partonic 2-surface (fixing the gauge)?

   On the other hand, the supersymplectic algebra (SSA) and extension of Kac-Moody algebras to light-like orbits of partonic 2-surfaces suggests strongly that functions of form $(t - z)^m v^n$ as basis associated with SSA and SKMAs must be allowed as basis at these 3-D light-like surfaces. These functions generate deformations of boundaries defining symmetries but the corresponding deformations in the interior of the preferred extremals are not expected to be of this form. Double loop algebra would not be lost but would have a nice separable form only at boundaries of CD and at light-like partonic orbits.

What can one conclude?

1. The general experience about the solutions of field equations conforms with this picture coded to the notion of Hamilton-Jacobi structure [K3]. Field equations and purely number theoretic conditions related to Minkowski signature force what might be called number theoretic spontaneous symmetry breaking. This symmetry breaking is analogous to a selection of single imaginary unit defining the analog of Kähler structure for M^4: this imaginary unit defines a new kind of $U(1)$ force in TGD explaining large scale breaking of CP,P, and T [L13]. This kind of selection occurs also for the quaternionic structure of CP_2 [L16].

2. The realistic form of analyticity condition abstractable from the properties known extremals seems to be following. For the Minkowskian space-time surfaces the complex coordinates of H are analytic functions of complex coordinates and of light-like coordinate assignable to space-time surface. These coordinates can be assigned to M^4 and define decomposition $M^4 = M^2 \times E^2$: this decomposition can be local but must be integrable (Hamilton-Jacobi structure). For Euclidian regions with 4-D CP_2 projection complex coordinate of E^2 is complex function of complex coordinates of CP_2 and M^2 light-like coordinate is function of real CP_2 coordinates and second light-like coordinate is constant.

3. The transition to Minkowskian signature by regarding M^4 as sub-space of complex-quaternionic M^4 does not respect the notion of quaternion analyticity in the naïvest sense. Both Euclidian and Minkowskian variants of quaternionic (associative) sub-manifold however makes sense as also co-quaternionic (co-associative) sub-manifold. An attractive hypothesis is that the geometric view about quaternionicity is consistent with the above view about analyticity. The known extremals are consistent with this form of analyticity. Analyticity in this sense should be consistent with the geometric quaternionicity of X^4 in Minkowskian signature and geometric co-quaternionicity in Euclidian signature.

4. The geometric form of quaternionicity (or associativity) requires that the associator $a(bc) - (ab)c$ for any 3 tangent space vectors vanishes. These conditions involve products of 3 partial derivates of embedding space coordinates. For co-associativity this holds true in the normal space. Again one must remember that these conditions might be needed only in M^8 but make sense also for H.

   One must be however cautious: quaternionicity (associativity) in M^8 in the geometric sense *need not* imply even the above realistic form of quaternion analyticity condition in M^8 and even less so in H: this however seems to be the case.

3.3.3 Can the known extremals satisfy the realistic form of quaternion-analyticity?

To test the consistency the realistic form of quaternion analyticity, at the level of M^8 or even H, the best thing to do is to look whether quaternion analyticity is possible for the known extremals for the twistor lift of Kähler action.

Twistor lift drops away most vacuum extremals from consideration and leaves only minimal surfaces. The surviving vacuum extremals include CP_2 type extremals with light-like geodesic rather than arbitrary light-like curve as M^4 projection. Vacuum extremals expressible as graph of map from M^4 to a Lagrangian sub-manifold of CP_2 remain in the spectrum only if they are also minimal surfaces: this kind of minimal surfaces are known to exist.

Massless extremals (MEs) with 2-D CP_2 projection remain in the spectrum. Cosmic strings of form $X^2 \times Y^2 \subset M^4 \times CP_2$ such that X^2 is string world sheet (minimal surface) and Y^2 complex sub-manifold of CP_2 are extremals of both Kähler action and volume term. One can also check whether Hamilton-Jacobi structure of M^4 and of Minkowskian space-time regions and complex structure of CP_2 have natural counterparts in the quaternion-analytic framework.

1. Consider first cosmic strings. In this case the quaternionic-analytic map from $X^4 = X^2 \times Y^2$ to $M^4 \times CP_2$ with octonion structure would be map X^4 to 2-D string world sheet in M^4 and Y^2 to 2-D complex manifold of CP_2. This could be achieved by using the linear variant of CRF condition. The map from X^4 to M^4 would reduce to ordinary hyper-analytic map from X^2 with hyper-complex coordinate to M^4 with hyper-complex coordinates just as in string models. The map from X^4 to CP_2 would reduce to an ordinary analytic map from X^2 with complex coordinates. One would not leave the realm of string models.

2. For the simplest massless extremals (MEs) CP_2 coordinates are arbitrary functions of light-like coordinate $u = k \cdot m$, k constant light-like vector, and of $v = \epsilon \cdot m$, ϵ - a polarization vector orthogonal to k. The interpretation as classical counterpart of photon or Bose-Einstein condensate of photons is obvious. There are good reasons to expect that this ansatz generalizes by replacing the variables u and v with coordinate along the light-like and space-like coordinate lines of Hamilton-Jacobi structure [K3]. The non-geodesic motion of photons with light-velocity and variation of the polarization direction would be due to the interactions with the space-time sheet to which it is topologically condensed.

   Now space-time surface would have naturally M^4 coordinates and the map $M^4 \rightarrow M^4$ would be just identity map satisfying the radial CRF condition. Can one understand CP_2 coordinates in terms of the realistic form of quaternion- analyticity? The dependence of CP_2 coordinates on $u = t - x$ only can be formulated as CFR condition $\partial_{\overline{u}} s^k = 0$ and this could is expected to generalize in the formulation using the geometric representation of quaternionic imaginary units at both sides. The dependence on light-light coordinate u follows from the translationally invariant CRF condition.

The dependence on the real coordinate $v = t - z$ does not conform with the proposed ansatz since the dependence is naturally on complex coordinate w assignable to the polarization plane of form $z = f(w)$. This would give dependence on 2 transversal coordinates and CP_2 projection would be 3-D rather than 2-D. One can of course ask whether this dependence is actually present for preferred extremals? Could the polarization vector be complex local polarization vector orthogonal to the light-like vector? In quantum theory complex polarization vectors are used routinely and become oscillator operators in second quantization and in TGD Universe MEs indeed serve as space-time correlates for photons or their BE condensates.

If this picture makes sense, one must modify the ansatz for the preferred extremals with Minkowskian signature. The E^4 and coordinates and perhaps even real CP_2 coordinates can depend on light-like coordinate u.

3. Vacuum extremals with Lagrangian manifold as (in the generic case 2-D) CP_2 projection survive if they are minimal surfaces. This property should guarantee the realistic form of quaternion analyticity. Hyper-quaternionic structure for space-time surface using Hamilton-Jacobi structure is the first guess. CP_2 should allow a quaternionic coordinate decomposing to a pair of complex coordinates such that second complex coordinate is constant for 2-D Lagrangian manifold and second parameterizes it. Any 2-D surface allows complex structure defined by the induced metric so that there are good hopes that these coordinates exist. The quaternion-analytic map would map in the most general case is trivial for both hypercomplex and complex coordinate of M^4 but the quaternionic Taylor coefficients reduce to real numbers to that the image is 2-D.

4. For CP_2 type vacuum extremals surviving as extremals the M^4 projection is light-like geodesic with $t + z = 0$ with suitable choice of light-like coordinates in M^2. $t - z$ can arbitrary function of CP_2 coordinates. Associativity of the normal space is the only possible option now.

   One can fix the coordinates of X^4 to be complex coordinates of CP_2 so that one gets rid of the degeneracy due to the choice of coordinates. M^4 allows hyper-quaternionic coordinates and Hamilton-Jacobi structures define different choices of hyper-quaternionic coordinates. Now the second light- like coordinate would vary along random light-like geodesics providing slicing of M^4 by 3-D surfaces. Hamilton-Jacobi structure defines at each point a plane $M^2(x)$ fixed by the light-like vector at the point and the 2-D orthogonal plane. In fact 4-D coordinate grid is defined.

5. In the naïve generalization CRF conditions are linear. Whether this is the case in the formulation using the geometric representation of the imaginary units is not clear since the quaternionic imaginary units depend on the vielbein of the induced 3-metric (note however that the SO(3) gauge rotation appearing in the conditions could allow to compensate for the change of the tensors in small deformations of the spaced-time surface). If linearity is real and not true only for small perturbations, one could have linear superpositions of different types of solutions, which looks strange. Could the superpositions describe perturbations of say cosmic strings and massless extremals?

6. According to [A12] both forms of the algebraic CRF conditions generalize to the octonionic situation and right multiplication of powers of octonion by Taylor coefficients plus linearity imply that there are no problems with associativity. This inspires several questions.

   Could octonion analytic maps of embedding space allow to construct new solutions from the existing ones? Could quaternion analytic maps applied at space-time level act as analogs of holomorphic maps and generalize conformal gauge invariance to 4-D context?

3.3.4 Quaternion analyticity and generalized conformal algebras

The realistic quaternionic analyticity should apply at the level of conformal algebras for conformal algebra is replaced with a direct sum of 2-D conformal and hyper-conformal algebra assignable to string world sheets and partonic 2-surfaces. This would conform with SH and the considerations above.

It is however too early to exclude the possibility that the powers z^n of conformal algebras are replaced by $u^m z^n$ ($u = t - z$ and $w = x + iy$) for symmetries restricted to the light-like boundaries of CD and to the light-like orbits of partonic 2-surfaces. This preferred form at boundaries would be essential for reducing degrees of freedom implied by SSA and SKMA gauge conditions. In the interior of space-time surfaces this simple form would be lost.

This would realize the Minkowskian analog of double loop algebras suggested by 4-dimensionality. This option is encouraged by the structure of super-symplectic algebra and generalization of Kac-Moody algebras for light-like orbits of partonic 2-surfaces. Again one must however remember that these algebras should have a realization at the level of M^8 but might not be necessary at the level of H.

1. The basic vision of quantum TGD is that string world sheets are replaced with 4-D surfaces and this forces a generalization of the notion of conformal invariance and one indeed obtains generalized conformal invariances for both the light-like boundaries of CD and for the light-like 3-surfaces defining partonic orbits as boundaries between Minkowskian and Euclidian space-time regions. One can very roughly say that string string world sheets parameterized by complex coordinate are replaced by space-time surfaces parameterized by two complex coordinates. Quaternion analyticity in the sense discussed would roughly conform with this picture.

2. The recent work with the Yangians [K12] and so called n-structures related to the categorification of TGD [K5] suggest that double loop algebras for which string world sheets are replaced with 4-D complex spaces. Quantum groups and Yangians assignable to Kac-Moody algebras rather than Lie algebras should be also central. Also double quantum groups depending on 2 parameters with so called elliptic R-matrix seem to be important. This physical intuition agrees with the general vision of Russian mathematician Igor Frenkel, who is one of the pioneers of quantum groups. For the article summarizing the work of Frenkel see `http://tinyurl.com/y7eego8c`). This article tells also about the work of Frenkel related to quaternion analyticity, which he sees to be of physical relevance but as a mathematicians is well aware of the fact that the non-commutativity of quaternions poses strong interpretation problems and means the loss of many nice properties of the ordinary analyticity.

3. The twistor lift of TGD suggest similar picture [K16, K4, L16]. The 6-D twistor space of space-time surface would be 6-surface in the product $T(M^4) \times T(CP_2)$ of geometric twistor spaces of M^4 and CP_2 and have induced twistor structure. The detailed analysis of this statement strongly suggests that data given at surfaces with dimension not higher than $D = 2$ should fix the preferred extremals. For the twistor lift action contains besides Kähler action also volume term. Asymptotic solutions are extremals of both Kähler action and minimal surfaces and all non-vacuum extremals of Kähler action are minimal surfaces so that the only change is that vacuum extremals of Kähler action must be restricted to be minimal surfaces.

The article about the work of Igor Frenkel (see `http://tinyurl.com/y7eego8c`) explains the general mathematics inspired vision about 3-levelled hierarchy of symmetries.

1. At the lowest level are Lie algebras. Gauge theories are prime example about this level.

2. At the second level loop algebras and quantum groups (defined as deformations of enveloping algebra of Lie algebra) and also Yangians. Loop algebras correspond to string models and quantum groups to TQFTs formulated at 3-D spaces.

3. At the third level are double loop algebras, quantum variants of loop algebras (also Yangians), and double quantum quantum groups - deformations of Lie algebras for which the R-matrix is elliptic function and depends on 2 complex parameters.

The conjecture of Frenkel (see `http://tinyurl.com/y7eego8c`) based on mathematical intuition is that these levels are actually the only ones. The motivation for this claim is 2-dimensionality making possible braiding and various quantum algebras. The set of poles for the R-matrix forms Abelian group with respect to addition in complex plane and can have rank equal to 0, 1, or (single pole, poles along line, lattice of poles). Higher ranks are impossible in $D = 2$.

In TGD framework physical intuition leads to a similar vision.

1. The dimension $D = 4$ for space-time surface and the choice $H = M^4 \times CP_2$ have both number theoretical and twistorial motivations [K12]. The replacement of point like particle with partonic 2-surface implies that TGD corresponds to the third level since loop algebras are replaced with their double loop analogs. 4-dimensionality makes also possible 2-braids and reconnections giving rise to a new kind of topological physics.

   The double loop group would represent the most dynamical level and its singly and doubly quantized variants correspond to a reduction in degrees of freedom, which one cannot exclude in TGD.

   The interesting additional aspect relates to the adelic physics [L14] implying a hierarchy of physics labelled by extensions of rationals. For cognitive representations the dynamics is discretized [K5]. Light-like 3-surfaces as partonic orbits are part of the picture and Chern-Simons term is naturally associated with them. TGD as almost topological QFT has been one of the guiding ideas in the evolution of TGD.

2. Double loop algebras represent unknown territory of mathematical physics. Igor Frenkel has also considered a possible realization of double loop algebras (see `http://tinyurl.com/y7eego8c`). He starts from the work of Mickelson (by the way, my custos in my thesis defence in 1982!) giving a realization of loop algebras: the idea is clearly motivated by WZW model which is 2-D conformal field theory with action containing a term associated with a 3-ball having 2-sphere as boundary.

   Mickelson starts from a circle represented as a boundary of a disk at which the physical states of CFT are realized. CFT is obtained by gluing together two disks with the boundary circles identified. The sphere in turn can be regarded as a boundary of a ball. The proposal of Frenkel is to complexify all these structures: circle becomes a Riemann surface, disk becomes 4-D manifold possessing complex structure in some sense, and 3-ball becomes 6-D complex manifold in some sense conjectured to be Calabi-Yau manifold.

3. The twistor lift of TGD leads to an analogous proposal. Circle is replaced with partonic 2-surfaces and string world sheets. 2-D complex surface is replaced with space-time region with complex structure or Hamilton-Jacobi structure [K3] and possessing twistor structure. 6-D Calabi-Yau manifold is replaced with the 6-D twistor space of space-time surface (sphere bundle over space-time surface) represented as 6-surface in 12-D Cartesian product $T(H) = T(M^4) \times T(CP_2)$ of the geometric twistor spaces of M^4 and CP_2.

   Twistor structure is induced and this is conjectured to determine the dynamics to be that for the preferred extremals of Kähler action plus volume term. This vision would generalize Penrose's original vision by eliminating gauge fields as primary dynamical variables and replacing there dynamics with the geometrodynamics of space-time surface.

3.3.5 Do isometry currents of preferred extremals satisfy Frobenius integrability conditions?

During the preparation of the book I learned that Agostino Prastaro [A6, A7] has done highly interesting work with partial differential equations, also those assignable to geometric variational principles such as Kähler action of its twistor lift in TGD. I do not understand the mathematical details but the key idea is a simple and elegant generalization of Thom's cobordism theory, and it is difficult to avoid the idea that the application of Prastaro's idea might provide insights about the preferred extremals, whose identification is now on rather firm basis.

One could also consider a definition of what one might call dynamical homotopy groups as a genuine characteristics of WCW topology. The first prediction is that the values of conserved classical Noether charges correspond to disjoint components of WCW. Could the natural topology in the parameter space of Noether charges zero modes of WCW metric) be p-adic and realize adelic physics at the level of WCW? An analogous conjecture was made on basis of spin glass analogy long time ago. Second surprise is that the only the 6 lowest dynamical homotopy/homology groups of WCW would be non-trivial. The Kähler structure of WCW suggests that only Π_0, Π_2, and Π_4 are non-trivial.

The interpretation of the analog of Π_1 as deformations of generalized Feynman diagrams with elementary cobordism snipping away a loop as a move leaving scattering amplitude invariant conforms with the number theoretic vision about scattering amplitude as a representation for a sequence of algebraic operation can be always reduced to a tree diagram. TGD would be indeed topological QFT: only the dynamical topology would matter.

A further outcome is an ansatz for generalizing the earlier proposal for preferred extremals and stating that non-vanishing conserved isometry currents satisfy Frobenius integrability conditions so that one could assign global coordinate with their flow lines. This ansatz looks very similar to the CRF conditions stating quaternion analyticity [L4].

3.3.6 Conclusions

To sum up, connections between different conjectures related to the preferred extremals - M^8-H duality, Hamilton-Jacobi structure, induced twistor space structure, quaternion-Kähler property and its Minkowskian counterpart, and perhaps even quaternion analyticity - albeit not in the naïve form , are clearly emerging. The underlying reason is strong form of GCI forced by the construction of WCW geometry and implying strong from of holography posing extremely powerful quantization conditions on the extremals of Kähler action in ZEO. Without the conformal gauge conditions the mutual inconsistency of these conjectures looks rather infeasible.

4 Octo-Twistors And Twistor Space

The basic problem of the twistor approach is that one cannot represent massive momenta in terms of twistors in an elegant manner. One can also consider generalization of the notion of spinor and twistor. I have proposed a possible representation of massive states based on the existence of preferred plane of M^2 in the basic definition of theory allowing to express four-momentum as one of two light-like momenta allowing twistor description. One could however ask whether some more elegant representation of massive M^4 momenta might be possible by generalizing the notion of twistor -perhaps by starting from the number theoretic vision.

The basic idea is obvious: in quantum TGD massive states in M^4 can be regarded as massless states in M^8 and $M^4 \times CP_2$ (recall M^8-H duality). One can therefore map any massive M^4 momentum to a light-like M^8 momentum and hope that this association could be made in a unique manner. One should assign to a massless 8-momentum an 8-dimensional spinor of fixed chirality. The spinor assigned with the light-like four-momentum is not unique without additional conditions. The existence of covariantly constant right-handed neutrino in CP_2 degrees generating the super-conformal symmetries could allow to eliminate the non-uniqueness. 8-dimensional twistor in M^8 would be a pair of this kind of spinors fixing the momentum of massless particle and the point through which the corresponding light-geodesic goes through: the set of these points forms 8-D light-cone and one can assign to each point a spinor. In $M^4 \times CP_2$ definitions makes also in the case of $M^4 \times CP_2$ and twistor space would also now be a lifting of the space of light-like geodesics.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the Kähler-Dirac gamma matrices both in M^8 and H.

The basic challenge is to achieve twistorial description of four-momenta or even $M^4 \times CP_2$ quantum numbers: this applies both to the momenta of fundamental fermions at the lines of generalized Feynman diagrams and to the massive incoming and outcoming states identified as their composites.

1. A rather attractive way to overcome the problem at the level of fermions propagating along the braid strands at the light-like orbits of partonic 2-surfaces relies on the assumption that generalized Feynman diagrammatics effectively reduces to a form in which all fermions in the propagator lines are massless although they can have non-physical helicity [K39]. One can use ordinary M^4 twistors. This is consistent with the idea that space-time surfaces are quaternionic sub-manifolds of octonionic embedding space.

2. Incoming and outgoing states are composites of massless fermions and not massless. They are however massless in 8-D sense. This suggests that they could be described using generalization of twistor formalism from M^4 to M^8 and even betterm to $M^4 \times CP_2$.

In the following two possible twistorializations are considered.

4.1 Two ways To Twistorialize Embedding Space

In the following the generalization of twistor formalism for M^8 or $M^4 \times CP_2$ will be considered in more detail. There are two options to consider.

1. For the first option one assigns to $M^4 \times CP_2$ twistor space as a product of corresponding twistor spaces $T(M_4) = CP_3$ and the flag-manifold $T(CP_2) = SU(3)/U(1) \times U(1)$ parameterizing the choices of quantization axes for $SU(3)$: $T_H = T(M^4) \times T(CP_2)$. Quite remarkably, M^4 and CP_2 are the only 4-D manifolds allowing twistor space with Kähler structure. The twistor space is 12-dimensional. The choice of quantization axis is certainly a physically well-definec operation so that $T(CP_2)$ has physical interpretation. If all observable physical states are color singlets situation becomes more complex. If one assumes QCC for color quantum numbers Y and I_3, then also the choice of color quantization axis is fixed at the level of Kähler action from the condition that Y and I_3 have classically their quantal values.

2. For the second option one generalizes the usual construction for M^8 regarded as tangent space of $M^4 \times CP_2$ (unless one takes $M^8 - H$ duality seriously).

The tangent space option looks like follows.

1. One can map the points of M^8 to octonions. One can consider 2-component spinors with octonionic components and map points of M^8 light-cone to linear combinations of 2×2 Pauli sigma matrices but with octonionic components. By the same arguments as in the deduction of ordinary twistor space one finds that 7-D light-cone boundary is mapped to 7+8 D space since the octonionic 2-spinor/its conjugate can be multiplied/divided by arbitrary octonion without changing the light-like point. By standard argument this space extends to 8+8-D space. The points of M^8 can be identified as 8-D octonionic planes (analogs of complex sphere CP_1 in this space. An attractive identification is as octonionic projective space OP_2. Remarkably, octonions do not allow higher dimensional projective spaces.

2. If one assumes that the spinors are quaternionic the twistor space should have dimension 7+4+1=12. This dimension is same as for $M^4 \times CP_2$. Does this mean that quaternionicity assumption reduces $T(M^8) = OP_2$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$? Or does it yield 12-D space $G_2/U(1) \times U(1)$, which is also natural since G_2 has 2-D Cartan algebra? Number theoretical compactification would transform $T(M^8) = G_2/U(1) \times U(1)$ to $T(H) = CP_3 \times SU(3)/U(1) \times U(1)$. This would not be surprising since in $M^8 - H$-duality CP_2 parametrizes (hyper)quaternionic planes containing preferred plane M^2.

   Quaternionicity is certainly very natural in TGD framework. Quaternionicity for 8-momenta does not in general imply that they reduce to the observed M^4-momenta unless one identifies M^4 as one particular subspace of M^8. In $M^8 - H$ duality one in principle allows all choices of M^4: it is of course unclear whether this makes any physical difference. Color confinement could be interpreted as a reduction of M^8 momenta to M^4 momenta and would also allow the interpretational problems caused by the fact that CP_2 momenta are not possible.

3. Since octonions can be regarded as complexified quaternions with non-commuting imaginary unit, one can say that quaternionic spinors in M^8 are "real" and thus analogous to Majorana spinors. Similar interpretation applies at the level of H. Could one can interpret the quaternionicity condition for space-time surfaces and embedding space spinors as TGD analog of Majorana condition crucial in super string models? This would also be crucial for understanding supersymmetry in TGD sense.

4.2 Octotwistorialization Of M^8

Consider first the twistorialization in 4-D case. In M^4 one can map light-like momoment to spinors satisfying massless Dirac equation. General point m of M^4 can be mapped to a pair of massless spinors related by incidence relation defining the point m. The essential element of this association is that mass squared can be defined as determinant of the 2×2 matrix resulting in the assignment. Light-likeness is coded to the vanishing of the determinant implying that the spinors defining its rows are linearly independent. The reduction of M^4 inner product to determinant occurs because the 2×2 matrix can be regarded as a matrix representation of complexified quaternion. Massless means that the norm of a complexified quaternion defined as the product of q and its conjugate vanishes. Incidence relation $s_1 = xs_2$ relating point of M^4 and pair of spinors defining the corresponding twistor, can be interpreted in terms of product for complexified quaternions.

The generalization to the 8-D situation is straightforward: replace quaternions with octonions.

1. The transition to M^8 means the replacement of quaternions with octonions. Masslessness corresponds to the vanishing norm for complexified octonion (hyper-octonion).

2. One should assign to a massless 8-momentum an 8-dimensional spinor identifiable as octonion - or more precisely as hyper-octonion obtained by multiplying the imaginary part of ordinary octonion with commuting imaginary unit j and defining conjugation as a change of sign of j or that of octonionic imaginar units.

3. This leads to a generalization of the notion of twistor consisting of pair of massless octonion valued spinors (octonions) related by the incidence relation fixing the point of M^8. The incidence relation for Euclidian octonions says $s_1 = xs_2$ and can be interpreted in terms of triality for $SO(8)$ relating conjugate spinor octet to the product of vector octed and spinor octet. For Minkowskian subspace of complexified octonions light-like vectors and s_1 and s_2 can be taken light-like as octonions. Light like x can annihilate s_2.

The possibility to interpret M^8 as hyperoctonionic space suggests also the possibility to define the 8-D counterparts of sigma matrices to hyperoctonions to obtain a representation of sigma matrix algebra which is not a matrix representation. The mapping of gamma matrices to this representation allows to define a notion of hyper-quaternionicity in terms of the Kähler-Dirac gamma matrices both in M^8 and H.

4.3 Octonionicity, $SO(1,7)$, G_2, And Non-Associative Malcev Group

The symmetries assignable with octonions are rather intricate. First of all, octonions (their hyper-variants defining M^8) have SO(8) (SO(1, 7)) as isometries. $G_2 \subset SO(7)$ acts as automorphisms of octonions and $SO(1,7) \rightarrow G_2$ clearly means breaking of Lorentz invariance.

John Baez has described in a lucid manner G_2 geometrically (`http://tinyurl.com/ybd4lcpy`). The basic observation is that quaternionic sub-space is generated by two linearly independent imaginary units and by their product. By adding a fourth linearly independent imaginary unit, one can generated all octonions. From this and the fact that G_2 represents subgroup of $SO(7)$, one easily deduces that G_2 is 14-dimensional. The Lie algebra of G_2 corresponds to derivations of octonionic algebra as follows infinitesimally from the condition that the image of product is the product of images. The entire algebra $SO(8)$ is direct sum of G_2 and linear transformations generated by right and left multiplication by imaginary octonion: this gives $14 + 14 = 28 = D(SO(8))$. The subgroup $SO(7)$ acting on imaginary octonsions corresponds to the direct sum of derivations and adjoint transformations defined by commutation with imaginary octonions, and has indeed dimension $14 + 7 = 21$.

One can identify also a non-associative group-like structure.

1. In the case of octonionic spinors this group like structure is defined by the analog of phase multiplication of spinor generalizing to a multiplication with octonionic unit expressible as linear combinations of 8 octonionic imaginary units and defining 7-sphere plays appear as analog of automorphisms $o \rightarrow uou^{-1} = uou^*$.

   One can associate with these transformations a non-associative Lie group and Lie algebra like structures by defining the commutators just as in the case of matrices that is as $[a,b] = ab-ba$. One 7-D non-associative Lie group like structure with topology of 7-sphere S^7 whereas G_2 is 14-dimensional exceptional Lie group (having S^6 as coset space $S^6 = G_2/SU(3)$). This group like object might be useful in the treatment of octonionic twistors. In the case of quaternions one has genuine group acting as SO(3) rotations.

2. Octonionic gamma matrices allow to define as their commutators octonionic sigma matrices:

$$\Sigma_{kl} = \frac{i}{2}\left[\gamma_k, \gamma_l\right] \quad . \tag{4.1}$$

   This algebra is 14-dimensional thanks to the fact that octonionic gamma matrices are of form $\gamma_0 = \sigma_1 \otimes 1$, $\gamma_i = \sigma_2 \otimes e_i$. Due to the non-associativity of octonions this algebra does not satisfy Jacobi identity - as is easy to verify using Fano triangle - and is therefore not a genuine Lie-algebra. Therefore these sigma matrices do not define a representation of G_2 as I thought first.

   This algebra has decomposition $g = h + t$, $[h, t] \subset t$, $[t, t] \subset h$ characterizing for symmetric spaces. h is the 7-D algebra generated by Σ_{ij} and identical with the non-associative Malcev algebra generated by the commutators of octonionic units. The complement t corresponds to the generators Σ_{0i}. The algebra is clearly an octonionic non-associative analog fo $SO(1,7)$.

4.4 Octonionic Spinors In M^8 And Real Complexified-Quaternionic Spinors In H?

This above observations about the octonionic sigma matrices raise the problem about the octonionic representation of spinor connection. In $M^8 = M^4 \times E^4$ the spinor connection is trivial but for $M^4 \times CP_2$ not. There are two options.

1. Assume that octonionic spinor structure makes sense for M^8 only and spinor connection is trivial.

2. An alternative option is to identify M^8 as tangent space of $M^4 \times CP_2$ possessing quaternionic structure defined in terms of octonionic variants of gamma matrices. Should one replace sigma matrices appearing in spinor connection with their octonionic analogs to get a sigma matrix algebra which is pseudo Lie algebra. Or should one map the holonomy algebra of CP_2 spinor connection to a sub-algebra of $G_2 \subset SO(7)$ and define the action of the sigma matrices as ordinary matrix multiplication of octonions rather than octonionic multiplication? This seems to be possible formally.

   The replacement of sigma matrices with their octonionic counterparts seems to lead to weird looking results. Octonionic multiplication table implies that the electroweak sigma matrices associated with CP_2 tangent space reduce to M^4 sigma matrices so that the spinor connection is quaternionic. Furthermore, left-handed sigma matrices are mapped to zero so that only the neutral part of spinor connection is non-vanishing. This supports the view that only M^8 gamma matrices make sense and that Dirac equation in M^8 is just free massless Dirac equation leading naturally also to the octonionic twistorialization.

   One might think that distinction between different H-chiralities is difficult to make but it turns out that quarks and leptons can be identified as different components of 2-component complexified octonionic spinors.

The natural question is what associativization of octonions gives. This amounts to a condition putting the associator $a(bc) - (ab)c$ to zero. It is enough to consider octonionic imaginary units which are different. By using the decomposition of the octonionic algebra to quaternionic sub-algebra and its complement and general structure of structure constants, one finds that quaternionic sub-algebra remains as such but the products of all imaginary units in the complement with different imaginary units vanish. This means that the complement behaves effectively as 4-D flat space-gamma matrix algebra annihilated by the quaternionic sub-algebra whose imaginary part acts like Lie algebra of $SO(3)$.

4.5 What The Replacement Of $SO(7,1)$ Sigma Matrices With Octonionic Sigma Matrices Could Mean?

The basic implication of octonionization is the replacement of $SO(7,1)$ sigma matrices with octonionic sigma matrices. For M^8 this has no consequences since since spinor connection is trivial.

For $M^4 \times CP_2$ situation would be different since CP_2 spinor connection would be replaced with its octonionic variant. This has some rather unexpected consequences and suggests that one should not try to octonionize at the level of $M^4 \times CP_2$ but interpret gamma matrices as tensor products of quaternionic gamma matrices, which can be replaced with their matrix representations. There are however some rather intriguing observations which force to keep mind open.

4.5.1 Octonionic representation of 8-D gamma matrices

Consider first the representation of 8-D gamma matrices in terms of tensor products of 7-D gamma matrices and 2-D Pauli sigma matrices.

1. The gamma matrices are given by

$$\gamma^0 = 1 \times \sigma_1 \ , \ \gamma^i = \gamma^i \otimes \sigma_2 \ , \ i = 1,..,7 \ . \tag{4.2}$$

   7-D gamma matrices in turn can be expressed in terms of 6-D gamma matrices by expressing γ^7 as

$$\gamma^{7)}_{i+1} = \gamma^{6)}_i, i = 1,...,6 \ , \ \gamma^{7)}_1 = \gamma^{6)}_7 = \prod_{i=1}^{6} \gamma^{6)}_i \ . \tag{4.3}$$

2. The octonionic representation is obtained as

$$\gamma_0 = 1 \otimes \sigma_1 \ , \ \gamma_i = e_i \otimes \sigma_2 \ . \tag{4.4}$$

   where e_i are the octonionic units. $e_i^2 = -1$ guarantees that the M^4 signature of the metric comes out correctly. Note that $\gamma_7 = \prod \gamma_i$ is the counterpart for choosing the preferred octonionic unit and plane M^2.

3. The octonionic sigma matrices are obtained as commutators of gamma matrices:

$$\Sigma_{0i} = je_i \times \sigma_3 \ , \ \Sigma_{ij} = jf_{ij}{}^{k} e_k \otimes 1 \ . \tag{4.5}$$

   Here j is commuting imaginary unit. These matrices span G_2 algebra having dimension 14 and rank 2 and having imaginary octonion units and their conjugates as the fundamental representation and its conjugate. The Cartan algebra for the sigma matrices can be chosen to be Σ_{01} and Σ_{23} and belong to a quaternionic sub-algebra.

4. The lower dimension $D = 14$ of the non-associative version of sigma matrix algebra algebra means that some combinations of sigma matrices vanish. All left or right handed generators of the algebra are mapped to zero: this explains why the dimension is halved from 28 to 14. From the octonionic triangle expressing the multiplication rules for octonion units [A3] one finds $e_4e_5 = e_1$ and $e_6e_7 = -e_1$ and analogous expressions for the cyclic permutations of e_4, e_5, e_6, e_7. From the expression of the left handed sigma matrix $I_L^3 = \sigma_{23} + \sigma^{30}$ representing left handed weak isospin (see the Appendix about the geometry of CP_2 [K7]) one can conclude that this particular sigma matrix and left handed sigma matrices in general are mapped to zero. The quaternionic sub-algebra $SU(2)_L \times SU(2)_R$ is mapped to that for the rotation group $SO(3)$ since in the case of Lorentz group one cannot speak of a decomposition to left and right handed subgroups. The elements of the complement of the quaternionic sub-algebra are expressible in terms of Σ_{ij} in the quaternionic sub-algebra.

4.5.2 Some physical implications of the reduction of $SO(7,1)$ to its octonionic counterpart

The octonization of spinor connection of CP_2 has some weird physical implications forcing to keep mind to the possibility that the octonionic description even at the level of H might have something to do with reality.

1. If $SU(2)_L$ is mapped to zero only the right-handed parts of electro-weak gauge field survive octonionization. The right handed part is neutral containing only photon and Z^0 so that the gauge field becomes Abelian. Z^0 and photon fields become proportional to each other ($Z^0 \rightarrow sin^2(\theta_W)\gamma$) so that classical Z^0 field disappears from the dynamics, and one would obtain just electrodynamics.

2. The gauge potentials and gauge fields defined by CP_2 spinor connection are mapped to fields in $SO(2) \subset SU(2) \times U(1)$ in quaternionic sub-algebra which in a well-defined sense corresponds to M^4 degrees of freedom and gauge group becomes $SO(2)$ subgroup of rotation group of $E^3 \subset M^4$. This looks like catastrophe. One might say that electroweak interactions are transformed to gravimagnetic interactions.

3. In very optimistic frame of mind one might ask whether this might be a deeper reason for why electrodynamics is an excellent description of low energy physics and of classical physics. This is consistent with the fact that CP_2 coordinates define 4 field degrees of freedom so that single Abelian gauge field should be enough to describe classical physics. This would remove also the interpretational problems caused by the transitions changing the charge state of fermion induced by the classical W boson fields.

4. Interestingly, the condition that electromagnetic charge is well-defined quantum number for the modes of the induced spinor field for $X^4 \subset H$ leads to the proposal that the solutions of the Kähler-Dirac equation are localized to string world sheets in Minkowskian regions of space-time surface at least. For CP_2 type vacuum extremals one has massless Dirac and this allows only covariantly constant right-handed neutrino as solution. One has however only a piece of CP_2 (wormhole contact) so that holomorphic solutions annihilated by two complexified gamma matrices are possible in accordance with the conformal symmetries.

Can one assume non-trivial spinor connection in M^8?

1. The simplest option encouraged by the requirement of maximal symmetries is that it is absent. Massless 8-momenta would characterize spinor modes in M^8 and this would give physical justification for the octotwistors.

2. If spinor connection is present at all, it reduces essentially to Kähler connection having different couplings to quarks and leptons identifiable as components of octonionic 2-spinors. It should be $SO(4)$ symmetric and since CP_2 is instant one might argue that now one has also instanton that is self-dual U(1) gauge field in $E^4 \subset M^4 \times E^4$ defining Kähler form. One can loosely say that one has of constant electric and magnetic fields which are parallel to each other. The rotational symmetry in E^4 would break down to $SO(2)$.

3. Without spinor connection quarks and leptons are in completely symmetric position at the level of M^8: this is somewhat disturbing. The difference between quarks and leptons in H is made possible by the fact that CP_2 does not allow standard spinor structure. Now this problem is absent. I have also consider the possibility that only leptonic spinor chirality is allowed and quarks result via a kind of anyonization process allowing them to have fractional em charges (see `http://tinyurl.com/y93aerea`).

4. If the solutions of the Kähler Dirac equation in Minkowskian regions are localized to two surfaces identifiable as integrable distributions of planes $M^2(x)$ and characterized by a local light-like direction defining the direction of massless momentum, they are holomorphic (in the sense of hyper-complex numbers) such that the second complexified Kähler-Dirac gamma matrix annihilates the solution. Same condition makes sense also at the level of M^8 for

solutions restricted to string world sheets and the presence or absence of spinor connection does not affect the situation.

Does this mean that the difference between quarks and leptons becomes visible only at the embedding space level where ground states of super-conformal representations correspond to embedding space spinor harmonics which in CP_2 cm degrees are different for quarks and leptons?

4.5.3 Octo-spinors and their relation to ordinary embedding space spinors

Octo-spinors are identified as octonion valued 2-spinors with basis

$$\begin{aligned} \Psi_{L,i} &= e_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \\ \Psi_{q,i} &= e_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad . \end{aligned} \tag{4.6}$$

One obtains quark and lepton spinors and conjugation for the spinors transforms quarks to leptons. Note that octospinors can be seen as 2-dimensional spinors with components which have values in the space of complexified octonions.

The leptonic spinor corresponding to real unit and preferred imaginary unit e_1 corresponds naturally to the two spin states of the right handed neutrino. In quark sector this would mean that right handed U quark corresponds to the real unit. The octonions decompose as $1+1+3+\overline{3}$ as representations of $SU(3) \subset G_2$. The concrete representations are given by

$$\begin{array}{ll} \{1 \pm ie_1\} \ , & e_R \text{ and } \nu_R \text{ with spin 1/2 } , \\ \{e_2 \pm ie_3\} \ , & e_R \text{ and } \nu_L \text{ with spin -1/2 } , \\ \{e_4 \pm ie_5\} & e_L \text{ and } \nu_L \text{ with spin 1/2 } , \\ \{e_6 \pm ie_7\} & e_L \text{ and } \nu_L \text{ with spin 1/2 } . \end{array} \tag{4.7}$$

Instead of spin one could consider helicity. All these spinors are eigenstates of e_1 (and thus of the corresponding sigma matrix) with opposite values for the sign factor $\epsilon = \pm$. The interpretation is in terms of vectorial isospin. States with $\epsilon = 1$ can be interpreted as charged leptons and D type quarks and those with $\epsilon = -1$ as neutrinos and U type quarks. The interpretation would be that the states with vanishing color isospin correspond to right handed fermions and the states with non-vanishing SU(3) isospin (to be not confused with QCD color isospin) and those with non-vanishing SU(3) isospin to left handed fermions.

The importance of this identification is that it allows a unique map of the candidates for the solutions of the octonionic Kähler-Dirac equation to those of ordinary one. There are some delicacies involved due to the possibility to chose the preferred unit e_1 so that the preferred subspace M^2 can corresponds to a sub-manifold $M^2 \subset M^4$.

4.6 About The Twistorial Description Of Light-Likeness In 8-D Sense Using Octonionic Spinors

The twistor approach to TGD [K39] require that the expression of light-likeness of M^4 momenta in terms of twistors generalizes to 8-D case. The light-likeness condition for twistors states that the 2×2 matrix representing M^4 momentum annihilates a 2-spinor defining the second half of the twistor. The determinant of the matrix reduces to momentum squared and its vanishing implies the light-likeness. This should be generalized to a situation in one has M^4 and CP_2 twistor which are not light-like separately but light-likeness in 8-D sense holds true.

4.6.1 The case of $M^8 = M^4 \times E^4$

$M^8 - H$ duality [K37] suggests that it might be useful to consider first the twistorialiation of 8-D light-likeness first the simpler case of M^8 for which CP_2 corresponds to E^4. It turns out that octonionic representation of gamma matrices provide the most promising formulation.

In order to obtain quadratic dispersion relation, one must have 2×2 matrix unless the determinant for the 4×4 matrix reduces to the square of the generalized light-likeness condition.

1. The first approach relies on the observation that the 2×2 matrices characterizing four-momenta can be regarded as hyper-quaternions with imaginary units multiplied by a commuting imaginary unit. Why not identify space-like sigma matrices with hyper-octonion units?
2. The square of hyper-octonionic norm would be defined as the determinant of 4×4 matrix and reduce to the square of hyper-octonionic momentum. The light-likeness for pairs formed by M^4 and E^4 momenta would make sense.
3. One can generalize the sigma matrices representing hyper-quaternion units so that they become the 8 hyper-octonion units. Hyper-octonionic representation of gamma matrices exists ($\gamma_0 = \sigma_z \times 1$, $\gamma_k = \sigma_y \times I_k$) but the octonionic sigma matrices represented by octonions span the Lie algebra of G_2 rather than that of $SO(1,7)$. This dramatically modifies the physical picture and brings in also an additional source of non-associativity. Fortunately, the flatness of M^8 saves the situation.
4. One obtains the square of $p^2 = 0$ condition from the massless octonionic Dirac equation as vanishing of the determinant much like in the 4-D case. Since the spinor connection is flat for M^8 the hyper-octonionic generalization indeed works.

This is not the only possibility that I have by-passingly considered [K10].

1. Is it enough to allow the four-momentum to be complex? One would still have 2×2 matrix and vanishing of complex momentum squared meaning that the squares of real and imaginary parts are same (light-likeness in 8-D sense) and that real and imaginary parts are orthogonal to each other. Could E^4 momentum correspond to the imaginary part of four-momentum?
2. The signature causes the first problem: M^8 must be replaced with complexified Minkowski space M^4_c for to make sense but this is not an attractive idea although M^4_c appears as subspace of complexified octonions.
3. For the extremals of Kähler action Euclidian regions (wormhole contacts identifiable as deformations of CP_2 type vacuum extremals) give imaginary contribution to the four-momentum. Massless complex momenta and also color quantum numbers appear also in the standard twistor approach. Also this suggest that complexification occurs also in 8-D situation and is not the solution of the problem.

4.6.2 The case of $M^8 = M^4 \times CP_2$

What about twistorialization in the case of $M^4 \times CP_2$? The introduction of wave functions in the twistor space of CP_2 seems to be enough to generalize Witten's construction to TGD framework and that algebraic variant of twistors might be needed only to realize quantum classical correspondence. It should correspond to tangent space counterpart of the induced twistor structure of space-time surface, which should reduce effectively to 4-D one by quaternionicity of the space-time surface.

1. For $H = M^4 \times CP_2$ the spinor connection of CP_2 is not trivial and the G_2 sigma matrices are proportional to M^4 sigma matrices and act in the normal space of CP_2 and to M^4 parts of octonionic embedding space spinors, which brings in mind co-associativity. The octonionic charge matrices are also an additional potential source of non-associativity even when one has associativity for gamma matrices.

   Therefore the octonionic representation of gamma matrices in entire H cannot be physical. It is however equivalent with ordinary one at the boundaries of string world sheets, where induced gauge fields vanish. Gauge potentials are in general non-vanishing but can be gauge transformed away. Here one must be of course cautious since it can happen that gauge fields vanish but gauge potentials cannot be gauge transformed to zero globally: topological quantum field theories represent basic example of this.

2. Clearly, the vanishing of the induced gauge fields is needed to obtain equivalence with ordinary induced Dirac equation. Therefore also string world sheets in Minkowskian regions should have 1-D CP_2 projection rather than only having vanishing W fields if one requires that octonionic representation is equivalent with the ordinary one. For CP_2 type vacuum extremals electroweak charge matrices correspond to quaternions, and one might hope that one can avoid problems due to non-associativity in the octonionic Dirac equation. Unless this is the case, one must assume that string world sheets are restricted to Minkowskian regions. Also embedding space spinors can be regarded as octonionic (possibly quaternionic or co-quaternionic at space-time surfaces): this might force vanishing 1-D CP_2 projection.

   (a) Induced spinor fields would be localized at 2-surfaces at which they have no interaction with weak gauge fields: of course, also this is an interaction albeit very implicit one! This would not prevent the construction of non-trivial electroweak scattering amplitudes since boson emission vertices are essentially due to re-groupings of fermions and based on topology change.

   (b) One could even consider the possibility that the projection of string world sheet to CP_2 corresponds to CP_2 geodesic circle so that one could assign light-like 8-momentum to entire string world sheet, which would be minimal surface in $M^4 \times S^1$. This would mean enormous technical simplification in the structure of the theory. Whether the spinor harmonics of embedding space with well-defined M^4 and color quantum numbers can co-incide with the solutions of the induced Dirac operator at string world sheets defined by minimal surfaces remains an open problem.

   (c) String world sheets cannot be present inside wormhole contacts which have 4-D CP_2 projection so that string world sheets cannot carry vanishing induced gauge fields.

5 What Could Be The Origin Of Preferred P-Adic Primes And P-Adic Length Scale Hypothesis?

p-Adic mass calculations [K25] allow to conclude that elementary particles correspond to one or possible several preferred primes assigning p-adic effective topology to the real space-time sheets in discretization in some length scale range. TGD inspired theory of consciousness leads to the identification of p-adic physics as physics of cognition. The recent progress leads to the proposal that quantum TGD is adelic: all p-adic number fields are involved and each gives one particular view about physics. tgdquantum/tgdquantum Adelic approach [K19, K29] plus the view about evolution as emergence of increasingly complex extensions of rationals leads to a possible answer to th question of the title. The algebraic extensions of rationals are characterized by preferred rational primes, namely those which are ramified when expressed in terms of the primes of the extensions. These primes would be natural candidates for preferred p-adic primes. An argument relying on what I call weak form of NMP in turn allows to understand why primes near powers of 2 are preferred: as a matter of fact, also primes near powers of other primes are predicted to be favoured.

5.1 Earlier Attempts

How the preferred primes emerge in TGD framework? I have made several attempts to answer this question. As a matter fact, the question has been slightly different: what determines the p-adic prime assigned to elementary particle by p-adic mass calculations [K22]. The recent view assigns to particle entire adele but some p-adic number fields in it are different.

1. Classical non-determinism at space-time level for real space-time sheets could in some length scale range involving rational discretization for space-time surface itself or for parameters characterizing it as a preferred extremal correspond to the non-determinism of p-adic differential equations due to the presence of pseudo constants which have vanishing p-adic derivative. Pseudo- constants are functions depend on finite number of pinary digits of its arguments.

2. The quantum criticality of TGD [?] is suggested to be realized in in terms of infinite hierarchies of super-symplectic symmetry breakings in the sense that only a sub-algebra with conformal weights which are n-ples of those for the entire algebra act as conformal gauge symmetries [K34]. This might be true for all conformal algebras involved. One has fractal hierarchy since the sub-algebras in question are isomorphic: only the scale of conformal gauge symmetry increases in the phase transition increasing n. The hierarchies correspond to sequences of integers $n(i)$ such tht $n(i)$ divides $n(i+1)$. These hierarchies would very naturally correspond to hierarchies of inclusions of hyper-finite factors and $m(i) = n(i+1)/n(i)$ could correspond to the integer n characterizing the index of inclusion, which has value $n \geq 3$. Possible problem is that $m(i) = 2$ would not correspond to Jones inclusion. Why the scaling by power of two would be different? The natural question is whether the primes dividing $n(i)$ or $m(i)$ could define the preferred primes.

3. Negentropic entanglement corresponds to entanglement for which density matrix is projector [K23]. For n-dimensional projector any prime p dividing n gives rise to negentropic entanglement in the sense that the number theoretic entanglement entropy defined by Shannon formula by replacing p_i in $log(p_i) = log(1/n)$ by its p-adic norm $N_p(1/n)$ is negative if p divides n and maximal for the prime for which the dividing power of prime is largest power-of-prime factor of n. The identification of p-adic primes as factors of n is highly attractive idea. The obvious question is whether n corresponds to the integer characterizing a level in the hierarchy of conformal symmetry breakings.

4. The adelic picture about TGD led to the question whether the notion of unitarity could be generalized. S-matrix would be unitary in adelic sense in the sense that $P_m = (SS^\dagger)_{mm} = 1$ would generalize to adelic context so that one would have product of real norm and p-adic norms of P_m. In the intersection of the realities and p-adicities P_m for reals would be rational and if real and p-adic P_m correspond to the same rational, the condition would be satisfied. The condition that $P_m \leq 1$ seems however natural and forces separate unitary in each sector so that this options seems too tricky.

These are the basic ideas that I have discussed hitherto.

5.2 Could Preferred Primes Characterize Algebraic Extensions Of Rationals?

The intuitive feeling is that the notion of preferred prime is something extremely deep and the deepest thing I know is number theory. Does one end up with preferred primes in number theory? This question brought to my mind the notion of ramification of primes (see http://tinyurl.com/hddljlf) (more precisely, of prime ideals of number field in its extension), which happens only for special primes in a given extension of number field, say rationals. Could this be the mechanism assigning preferred prime(s) to a given elementary system, such as elementary particle? I have not considered their role earlier also their hierarchy is highly relevant in the number theoretical vision about TGD.

1. Stating it very roughly (I hope that mathematicians tolerate this language): As one goes from number field K, say rationals Q, to its algebraic extension L, the original prime ideals in the so called integral closure (see http://tinyurl.com/js6fpvr) over integers of K decompose to products of prime ideals of L (prime is a more rigorous manner to express primeness).

   Integral closure for integers of number field K is defined as the set of elements of K, which are roots of some monic polynomial with coefficients, which are integers of K and having the form $x^n + a_{n-1}x^{n-1} + ... + a_0$. The integral closures of both K and L are considered. For instance, integral closure of algebraic extension of K over K is the extension itself. The integral closure of complex numbers over ordinary integers is the set of algebraic numbers.

2. There are two further basic notions related to ramification and characterizing it. Relative discriminant is the ideal divided by all ramified ideals in K and relative different is the ideal of L divided by all ramified P_i:s. Note that te general ideal is analog of integer and these ideas represent the analogous of product of preferred primes P of K and primes P_i of L dividing them.

3. A physical analogy is provided by decomposition of hadrons to valence quarks. Elementary particles becomes composite of more elementary particles in the extension. The decomposition to these more elementary primes is of form $P = \prod P_i^{e(i)}$, where e_i is the ramification index - the physical analog would be the number of elementary particles of type i in the state (see `http://tinyurl.com/h9528pl`). Could the ramified rational primes could define the physically preferred primes for a given elementary system?

In TGD framework the extensions of rationals (see `http://tinyurl.com/h9528pl`) and p-adic number fields (see`http://tinyurl.com/zq22tvb`) are unavoidable and interpreted as an evolutionary hierarchy physically and cosmological evolution would have gradually proceeded to more and more complex extensions. One can say that string world sheets and partonic 2-surfaces with parameters of defining functions in increasingly complex extensions of prime emerge during evolution. Therefore ramifications and the preferred primes defined by them are unavoidable. For p-adic number fields the number of extensions is much smaller for instance for $p > 2$ there are only 3 quadratic extensions.

1. In p-adic context a proper definition of counterparts of angle variables as phases allowing definition of the analogs of trigonometric functions requires the introduction of algebraic extension giving rise to some roots of unity. Their number depends on the angular resolution. These roots allow to define the counterparts of ordinary trigonometric functions - the naïve generalization based on Taylors series is not periodic - and also allows to defined the counterpart of definite integral in these degrees of freedom as discrete Fourier analysis. For the simplest algebraic extensions defined by $x^n - 1$ for which Galois group is abelian are are unramified so that something else is needed. One has decomposition $P = \prod P_i^{e(i)}$, $e(i) = 1$, analogous to n-fermion state so that simplest cyclic extension does not give rise to a ramification and there are no preferred primes.

2. What kind of polynomials could define preferred algebraic extensions of rationals? Irreducible polynomials are certainly an attractive candidate since any polynomial reduces to a product of them. One can say that they define the elementary particles of number theory. Irreducible polynomials have integer coefficients having the property that they do not decompose to products of polynomials with rational coefficients. IT would be wrong to say that only these algebraic extensions can appear but there is a temptation to say that one can reduce the study of extensions to their study. One can even consider the possibility that string world sheets associated with products of irreducible polynomials are unstable against decay to those characterize irreducible polynomials.

3. What can one say about irreducible polynomials? Eisenstein criterion (see `http://tinyurl.com/47kxjz` states following. If $Q(x) = \sum_{k=0,..,n} a_k x^k$ is n:th order polynomial with integer coefficients and with the property that there exists at least one prime dividing all coefficients a_i except a_n and that p^2 does not divide a_0, then Q is irreducible. Thus one can assign one or more preferred primes to the algebraic extension defined by an irreducible polynomial Q of this kind - in fact any polynomial allowing ramification. There are also other kinds of irreducible polynomials since Eisenstein's condition is only sufficient but not necessary.

4. Furthermore, in the algebraic extension defined by Q, the prime ideals P having the above mentioned characteristic property decompose to an n :th power of single prime ideal P_i: $P = P_i^n$. The primes are maximally/completely ramified. The physical analog $P = P_0^n$ is Bose-Einstein condensate of n bosons. There is a strong temptation to identify the preferred primes of irreducible polynomials as preferred p-adic primes.

   A good illustration is provided by equations $x^2 + 1 = 0$ allowing roots $x_\pm = \pm i$ and equation $x^2 + 2px + p = 0$ allowing roots $x_\pm = -p \pm \sqrt{p}p - 1$. In the first case the ideals associated with $\pm i$ are different. In the second case these ideals are one and the same since $x_+ == -x_- + p$: hence one indeed has ramification. Note that the first example represents also an example of irreducible polynomial, which does not satisfy Eisenstein criterion. In more general case the n conditions on defined by symmetric functions of roots imply that the ideals are one and same when Eisenstein conditions are satisfied.

5. What does this mean in p-adic context? The identity of the ideals can be stated by saying $P = P_0^n$ for the ideals defined by the primes satisfying the Eisenstein condition. Very loosely one can say that the algebraic extension defined by the root involves n:th root of p-adic prime p. This does not work! Extension would have a number whose n:th power is zero modulo p. On the other hand, the p-adic numbers of the extension modulo p should be finite field but this would not be field anymore since there would exist a number whose n:th power vanishes. The algebraic extension simply does not exist for preferred primes. The physical meaning of this will be considered later.

6. What is so nice that one could readily construct polynomials giving rise to given preferred primes. The complex roots of these polymials could correspond to the points of partonic 2-surfaces carrying fermions and defining the ends of boundaries of string world sheet. It must be however emphasized that the form of the polynomial depends on the choices of the complex coordinate. For instance, the shift $x \to x + 1$ transforms $(x^n - 1)/(x - 1)$ to a polynomial satisfying the Eisenstein criterion. One should be able to fix allowed coordinate changes in such a way that the extension remains irreducible for all allowed coordinate changes.

   Already the integral shift of the complex coordinate affects the situation. It would seem that only the action of the allowed coordinate changes must reduce to the action of Galois group permuting the roots of polynomials. A natural assumption is that the complex coordinate corresponds to a complex coordinate transforming linearly under subgroup of isometries of the embedding space.

In the general situation one has $P = \prod P_i^{e(i)}$, $e(i) \geq 1$ so that aso now there are prefered primes so that the appearance of preferred primes is completely general phenomenon.

5.3 A Connection With Langlands Program?

In Langlands program (`http://tinyurl.com/ycej7s43`,`RecentAdvancesinLanglandsprogram`) [A9, A8] the great vision is that the n-dimensional representations of Galois groups G characterizing algebraic extensions of rationals or more general number fields define n-dimensional adelic representations of adelic Lie groups, in particular the adelic linear group $Gl(n, A)$. This would mean that it is possible to reduce these representations to a number theory for adeles. This would be highly relevant in the vision about TGD as a generalized number theory. I have speculated with this possibility earlier (`http://tinyurl.com/y9ee3lk6`) [K19] but the mathematics is so horribly abstract that it takes decade before one can have even hope of building a rough vision.

One can wonder whether the irreducible polynomials could define the preferred extensions K of rationals such that the maximal abelian extensions of the fields K would in turn define the adeles utilized in Langlands program. At least one might hope that everything reduces to the maximally ramified extensions.

At the level of TGD string world sheets with parameters in an extension defined by an irreducible polynomial would define an adele containing various p-adic number fields defined by the primes of the extension. This would define a hierarchy in which the prime ideals of previous level would decompose to those of the higher level. Each irreducible extension of rationals would correspond to some physically preferred p-adic primes.

It should be possible to tell what the preferred character means in terms of the adelic representations. What happens for these representations of Galois group in this case? This is known.

1. For Galois extensions ramification indices are constant: $e(i) = e$ and Galois group acts transitively on ideals P_i dividing P. One obtains an n-dimensional representation of Galois group. Same applies to the subgroup of Galois group G/I where I is subgroup of G leaving P_i invariant. This group is called inertia group. For the maximally ramified case G maps the ideal P_0 in $P = P_0^n$ to itself so that $G = I$ and the action of Galois group is trivial taking P_0 to itself, and one obtains singlet representations.

2. The trivial action of Galois group looks like a technical problem for Langlands program and also for TGD unless the singletness of P_i under G has some physical interpretation. One possibility is that Galois group acts as like a gauge group and here the hierarchy of sub-algebras of super-symplectic algebra labelled by integers n is highly suggestive. This raises

obvious questions. Could the integer n characterizing the sub-algebra of super-symplectic algebra acting as conformal gauge transformations, define the integer defined by the product of ramified primes? P_0^n brings in mind the n conformal equivalence classes which remain invariant under the conformal transformations acting as gauge transformations. . Recalling that relative discriminant is an of K ideal divisible by ramified prime ideals of K, this means that n would correspond to the relative discriminant for $K = Q$. Are the preferred primes those which are "physical" in the sense that one can assign to the states satisfying conformal gauge conditions?

If the Galois group corresponds to gauge symmetries for these primes, it is physically natural that the p-adic algebraic extension does not exists and that p-adic variant of the Galois group is absent. Nothing is lost from cognition since there is nothing to cognize!

5.4 What Could Be The Origin Of P-Adic Length Scale Hypothesis?

The argument would explain the existence of preferred p-adic primes. It does not yet explain p-adic length scale hypothesis [K28, K22] stating that p-adic primes near powers of 2 are favored. A possible generalization of this hypothesis is that primes near powers of prime are favored. There indeed exists evidence for the realization of 3-adic time scale hierarchies in living matter [?] (`http://tinyurl.com/jbh9m27`) and in music both 2-adicity and 3-adicity could be present, this is discussed in TGD inspired theory of music harmony and genetic code [K33].

The weak form of NMP might come in rescue here.

1. Entanglement negentropy for a negentropic entanglement [K23] characterized by n-dimensional projection operator is the $log(N_p(n)$ for some p whose power divides n. The maximum negentropy is obtained if the power of p is the largest power of prime divisor of p, and this can be taken as definition of number theoretic entanglement negentropy. If the largest divisor is p^k, one has $N = k \times log(p)$. The entanglement negentropy per entangled state is $N/n = klog(p)/n$ and is maximal for $n = p^k$. Hence powers of prime are favoured which means that p-adic length scale hierarchies with scales coming as powers of p are negentropically favored and should be generated by NMP. Note that $n = p^k$ would define a hierarchy of $h_{eff}/h = p^k$. During the first years of h_{eff} hypothesis I believe that the preferred values obey $h_{eff} = r^k$, r integer not far from $r = 2^{11}$. It seems that this belief was not totally wrong.

2. If one accepts this argument, the remaining challenge is to explain why primes near powers of two (or more generally p) are favoured. $n = 2^k$ gives large entanglement negentropy for the final state. Why primes $p = n_2 = 2^k - r$ would be favored? The reason could be following. $n = 2^k$ corresponds to $p = 2$, which corresponds to the lowest level in p-adic evolution since it is the simplest p-adic topology and farthest from the real topology and therefore gives the poorest cognitive representation of real preferred extremal as p-adic preferred extermal (Note that $p = 1$ makes formally sense but for it the topology is discrete).

3. Weak form of NMP [K23, K41] suggests a more convincing explanation. The density matrix of the state to be reduced is a direct sum over contributions proportional to projection operators. Suppose that the projection operator with largest dimension has dimension n. Strong form of NMP would say that final state is characterized by n-dimensional projection operator. Weak form of NMP allows free will so that all dimensions $n - k$, $k = 0, 1, ...n - 1$ for final state projection operator are possible. 1-dimensional case corresponds to vanishing entanglement negentropy and ordinary state function reduction isolating the measured system from external world.

4. The negentropy of the final state per state depends on the value of k. It is maximal if $n - k$ is power of prime. For $n = 2^k = M_k + 1$, where M_k is Mersenne prime $n - 1$ gives the maximum negentropy and also maximal p-adic prime available so that this reduction is favoured by NMP. Mersenne primes would be indeed special. Also the primes $n = 2^k - r$ near 2^k produce large entanglement negentropy and would be favored by NMP.

5. This argument suggests a generalization of p-adic length scale hypothesis so that $p = 2$ can be replaced by any prime.

This argument together with the hypothesis that preferred prime is ramified would correlate the character of the irreducible extension and character of super-conformal symmetry breaking. The integer n characterizing super-symplectic conformal sub-algebra acting as gauge algebra would depends on the irreducible algebraic extension of rational involved so that the hierarchy of quantum criticalities would have number theoretical characterization. Ramified primes could appear as divisors of n and n would be essentially a characteristic of ramification known as discriminant. An interesting question is whether only the ramified primes allow the continuation of string world sheet and partonic 2-surface to a 4-D space-time surface. If this is the case, the assumptions behind p-adic mass calculations would have full first principle justification.

5.5 A Connection With Infinite Primes?

Infinite primes are one of the mathematical outcomes of TGD [K36]. There are two kinds of infinite primes. There are the analogs of free many particle states consisting of fermions and bosons labelled by primes of the previous level in the hierarchy. They correspond to states of a supersymmetric arithmetic quantum field theory or actually a hierarchy of them obtained by a repeated second quantization of this theory. A connection between infinite primes representing bound statesandirreducible polynomials is highly suggestive.

1. The infinite prime representing free many-particle state decomposes to a sum of infinite part and finite part having no common finite prime divisors so that prime is obtained. The infinite part is obtained from "fermionic vacuum" $X = \prod_k p_k$ by dividing away some fermionic primes p_i and adding their product so that one has $X \to X/m + m$, where m is square free integer. Also $m = 1$ is allowed and is analogous to fermionic vacuum interpreted as Dirac sea without holes. X is infinite prime and pure many-fermion state physically. One can add bosons by multiplying X with any integers having no common denominators with m and its prime decomposition defines the bosonic contents of the state. One can also multiply m by any integers whose prime factors are prime factors of m.

2. There are also infinite primes, which are analogs of bound states and at the lowest level of the hierarchy they correspond to irreducible polynomials $P(x)$ with integer coefficients. At the second levels the bound states would naturally correspond to irreducible polynomials $P_n(x)$ with coefficients $Q_k(y)$, which are infinite integers at the previous level of the hierarchy.

3. What is remarkable that bound state infinite primes at given level of hierarchy would define maximally ramified algebraic extensions at previous level. One indeed has infinite hierarchy of infinite primes since the infinite primes at given level are infinite primes in the sense that they are not divisible by the primes of the previous level. The formal construction works as such. Infinite primes correspond to polynomials of single variable at the first level, polynomials of two variables at second level, and so on. Could the Langlands program could be generalized from the extensions of rationals to polynomials of complex argument and that one would obtain infinite hierarchy?

4. Infinite integers in turn could correspond to products of irreducible polynomials defining more general extensions. This raises the conjecture that infinite primes for an extension K of rationals could code for the algebraic extensions of K quite generally. If infinite primes correspond to real quantum states they would thus correspond the extensions of rationals to which the parameters appearing in the functions defining partonic 2-surfaces and string world sheets.

   This would support the view that partonic 2-surfaces associated with algebraic extensions defined by infinite integers and thus not irreducible are unstable against decay to partonic 2-surfaces which corresponds to extensions assignable to infinite primes. Infinite composite integer defining intermediate unstable state would decay to its composites. Basic particle physics phenomenology would have number theoretic analog and even more.

5. According to Wikipedia, Eisenstein's criterion (`http://tinyurl.com/47kxjz`) allows generalization and what comes in mind is that it applies in exactly the same form also at the higher levels of the hierarchy. Primes would be only replaced with prime polynomials and

the there would be at least one prime polynomial $Q(y)$ dividing the coefficients of $P_n(x)$ except the highest one such that its square would not divide P_0. Infinite primes would give rise to an infinite hierarchy of functions of many complex variables. At first level zeros of function would give discrete points at partonic 2-surface. At second level one would obtain 2-D surface: partonic 2-surfaces or string world sheet. At the next level one would obtain 4-D surfaces. What about higher levels? Does one obtain higher dimensional objects or something else. The union of n 2-surfaces can be interpreted also as $2n$-dimensional surface and one could think that the hierarchy describes a hierarchy of unions of correlated partonic 2-surfaces. The correlation would be due to the preferred extremal property of Kähler action.

One can ask whether this hierarchy could allow to generalize number theoretical Langlands to the case of function fields using the notion of prime function assignable to infinite prime. What this hierarchy of polynomials of arbitrary many complex arguments means physically is unclear. Do these polynomials describe many-particle states consisting of partonic 2-surface such that there is a correlation between them as sub-manifolds of the same space-time sheet representing a preferred extremals of Kähler action?

This would suggest strongly the generalization of the notion of p-adicity so that it applies to infinite primes.

1. This looks sensible and maybe even practical! Infinite primes can be mapped to prime polynomials so that the generalized p-adic numbers would be power series in prime polynomial - Taylor expansion in the coordinate variable defined by the infinite prime. Note that infinite primes (irreducible polynomials) would give rise to a hierarchy of preferred coordinate variables. In terms of infinite primes this expansion would require that coefficients are smaller than the infinite prime P used. Are the coefficients lower level primes? Or also infinite integers at the same level smaller than the infinite prime in question? This criterion makes sense since one can calculate the ratios of infinite primes as real numbers.

2. I would guess that the definition of infinite-P p-adicity is not a problem since mathematicians have generalized the number theoretical notions to such a level of abstraction much above of a layman like me. The basic question is how to define p-adic norm for the infinite primes (infinite only in real sense, p-adically they have unit norm for all lower level primes) so that it is finite.

3. There exists an extremely general definition of generalized p-adic number fields (see `http://tinyurl.com/y5zreeg`). One considers Dedekind domain D, which is a generalization of integers for ordinary number field having the property that ideals factorize uniquely to prime ideals. Now D would contain infinite integers. One introduces the field E of fractions consisting of infinite rationals.

   Consider element e of E and a general fractional ideal eD as counterpart of ordinary rational and decompose it to a ratio of products of powers of ideals defined by prime ideals, now those defined by infinite primes. The general expression for the p-adic norm of x is $x^{-ord(P)}$, where n defines the total number of ideals P appearing in the factorization of a fractional ideal in E: this number can be also negative for rationals. When the residue field is finite (finite field G(p,1) for p-adic numbers), one can take c to the number of its elements ($c = p$ for p-adic numbers.

   Now it seems that this number is not finite since the number of ordinary primes smaller than P is infinite! But this is not a problem since the topology for completion does not depend on the value of c. The simple infinite primes at the first level (free many-particle states) can be mapped to ordinary rationals and q-adic norm suggests itself: could it be that infinite-P p-adicity corresponds to q-adicity discussed by Khrennikov [A5]. Note however that q-adic numbers are not a field.

Finally a loosely related question. Could the transition from infinite primes of K to those of L takes place just by replacing the finite primes appearing in infinite prime with the decompositions? The resulting entity is infinite prime if the finite and infinite part contain no common prime divisors in L. This is not the case generally if one can have primes P_1 and P_2 of K having common divisors as primes of L: in this case one can include P_1 to the infinite part of infinite prime and P_2 to finite part.

6 More About Physical Interpretation Of Algebraic Extensions Of Rationals

The number theoretic vision has begun to show its power. The basic hierarchies of quantum TGD would reduce to a hierarchy of algebraic extensions of rationals and the parameters - such as the degrees of the irreducible polynomials characterizing the extension and the set of ramified primes (see `http://tinyurl.com/hddljlf`) - would characterize quantum criticality and the physics of dark matter as large h_{eff} phases. The value of $h_{eff}/h = n$ would naturally correspond to the order of the Galois group of the extension.

The conjecture is that preferred p-adic primes correspond to ramified primes for extensions of rationals for which especially many number theoretic discretizations of the space-time surfaces allow strong form of holography as an algebraic continuation of string world sheets to space-time surfaces. The generalization of the p-adic length scale hypothesis as a prediction of NMP is another conjecture. What remains to be shown that the primes predicted by generalization p-adic length scale hypothesis indeed are preferred primes in the proposed sense.

By strong form of holography the parameters characterizing string world sheets and partonic 2-surfaces serve as WCW coordinates. By various conformal invariances, one expects that the parameters correspond to conformal moduli, which means a huge simplification of quantum TGD since the mathematical apparatus of superstring theories becomes available and number theoretical vision can be realized. Scattering amplitudes can be constructed for a given algebraic extension and continued to various number fields by continuing the parameters which are conformal moduli and group invariants characterizing incoming particles.

There are many un-answered and even un-asked questions.

1. How the new degrees of freedom assigned to the n-fold covering defined by the space-time surface pop up in the number theoretic picture? How the connection with preferred primes emerges?

2. What are the precise physical correlates of the parameters characterizing the algebraic extension of rationals? Note that the most important extension parameters are the degree of the defining polynomial and ramified primes.

6.1 Some Basic Notions

Some basic information about extensions are in order. I emphasize that I am not a specialist.

6.1.1 Basic facts

The algebraic extensions of rationals are determined by roots of polynomials. Polynomials be decomposed to products of irreducible polynomials, which by definition do not contain factors which are polynomials with rational coefficients. These polynomials are characterized by their degree n, which is the most important parameter characterizing the algebraic extension.

One can assign to the extension primes and integers - or more precisely, prime and integer ideals. Integer ideals correspond to roots of monic polynomials $P_n(x) = x^n + ..a_0$ in the extension with integer coefficients. Clearly, for $n = 0$ (trivial extension) one obtains ordinary integers. Primes as such are not a useful concept since roots of unity are possible and primes which differ by a multiplication by a root of unity are equivalent. It is better to speak about prime ideals rather than primes.

Rational prime p can be decomposed to product of powers of primes of extension and if some power is higher than one, the prime is said to be ramified and the exponent is called ramification index. Eisenstein's criterion (see `http://tinyurl.com/47kxjz` states that any polynomial $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + ...a_1x + a_0$ for which the coefficients a_i, $i < n$ are divisible by p and a_0 is not divisible by p^2 allows p as a maximally ramified prime. The corresponding prime ideal is n:th power of the prime ideal of the extensions (roughly n:th root of p). This allows to construct endless variety of algebraic extensions having given primes as ramified primes.

Ramification is analogous to criticality. When the gradient potential function $V(x)$ depending on parameters has multiple roots, the potential function becomes proportional a higher power of

$x - x_0$. The appearance of power is analogous to appearance of higher power of prime of extension in ramification. This gives rise to cusp catastrophe. In fact, ramification is expected to be number theoretical correlate for the quantum criticality in TGD framework. What this precisely means at the level of space-time surfaces, is the question.

6.1.2 Galois group as symmetry group of algebraic physics

I have proposed long time ago that Galois group (see `http://tinyurl.com/h9528pl`) acts as fundamental symmetry group of quantum TGD and even made clumsy attempt to make this idea more precise in terms of the notion of number theoretic braid. It seems that this notion is too primitive: the action of Galois group must be realized at more abstract level and WCW provides this level.

First some facts (I am not a number theory professional, as the professional reader might have already noticed!).

1. Galois group acting as automorphisms of the field extension (mapping products to products and sums to sums and preserves norm) characterizes the extension and its elements have maximal order equal to n by algebraic n-dimensionality. For instance, for complex numbers Galois group acs as complex conjugation. Galois group has natural action on prime ideals of extension mapping them to each other and preserving the norm determined by the determinant of the linear map defined by the multiplication with the prime of extension. For instance, for the quadratic extension $Q(\sqrt{5})$ the norm is $N(x + \sqrt{5}y) = x^2 - 5y^2$: not that number theory leads to Minkowkian metric signatures naturally. Prime ideals combine to form orbits of Galois group.

2. Since Galois group leaves the rational prime p invariant, the action must permute the primes of extension in the product representation of p. For ramified primes the points of the orbit of ideal degenerate to single ideal. This means that primes and quite generally, the numbers of extension, define orbits of the Galois group.

Galois group acts in the space of integers or prime ideals of the algebraic extension of rationals and it is also physically attractive to consider the orbits defined by ideals as preferred geometric structures. If the numbers of the extension serve as parameters characterizing string world sheets and partonic 2-surfaces, then the ideals would naturally define subsets of the parameter space in which Galois group would act.

The action of Galois group would leave the space-time surface invariant if the sheets co-incide at ends but permute the sheets. Of course, the space-time sheets permuted by Galois group need not co-incide at ends. In this case the action need not be gauge action and one could have non-trivial representations of the Galois group. In Langlands correspondence these representation relate to the representations of Lie group and something similar might take place in TGD as I have indeed proposed.

The value of effective Planck constant $h_{eff}/h = n$ corresponds to the number of sheets of some kind of covering space defined by the space-time surface. The discretization of the space-time surface identified as a monadic manifold [L10] with embedding space preferred coordinates in extension of rationals defining the adele has Galois group of extension as a group of symmetries permuting the sheets of the covering group. Therefore $n = h_{eff}/h$ would naturally correspond to the dimension of the extension dividing the order of its Galois group. Dark matter in TGD sense would correspond to number theoretic physics.

Remark: Strong form of holography supports also the vision about quaternionic generalization of conformal invariance implying that the adelic space-time surface can be constructed from the data associated with functions of two complex variables, which in turn reduce to functions of single variable.

If this picture is correct, it is possible to talk about quantum amplitudes in the space defined by the numbers of extension and restrict the consideration to prime ideals or more general integer ideals.

1. These number theoretical wave functions are physical if the parameters characterizing the 2-surface belong to this space. One could have purely number theoretical quantal degrees

of freedom assignable to the hierarchy of algebraic extensions and these discrete degrees of freedom could be fundamental for living matter and understanding of consciousness.

2. The simplest assumption that Galois group acts as a gauge group when the ends of sheets co-incide at boundaries of CD seems however to destroy hopes about non-trivial number theoretical physics but this need not be the case. Physical intuition suggests that ramification somehow saves the situation and that the non-trivial number theoretic physics could be associated with ramified primes assumed to define preferred p-adic primes.

6.2 How New Degrees Of Freedom Emerge For Ramified Primes?

How the new discrete degrees of freedom appear for ramified primes?

1. The space-time surfaces defining singular coverings are n-sheeted in the interior. At the ends of the space-time surface at boundaries of CD however the ends co-incide. This looks very much like a critical phenomenon.

   Hence the idea would be that the end collapse can occur only for the ramified prime ideals of the parameter space - ramification is also a critical phenomenon - and means that some of the sheets or all of them co-incide. Thus the sheets would co-incide at ends only for the preferred p-adic primes and give rise to the singular covering and large h_{eff}. End-collapse would be the essence of criticality! This would occur, when the parameters defining the 2-surfaces are in a ramified prime ideal.

2. Even for the ramified primes there would be n distinct space-time sheets, which are regarded as physically distinct. This would support the view that besides the space-like 3-surfaces at the ends the full 3-surface must include also the light-like portions connecting them so that one obtains a closed 3-surface. The conformal gauge equivalence classes of the light-like portions would give rise to additional degrees of freedom. In space-time interior and for string world sheets they would become visible.

   For ramified primes n distint 3-surfaces would collapse to single one but the n discrete degrees of freedom would be present and particle would obtain them. I have indeed proposed number theoretical second quantization assigning fermionic Clifford algebra to the sheets with n oscillator operators. Note that this option does not require Galois group to act as gauge group in the general case. This number theoretical second quantization might relate to the realization of Boolean algebra suggested by weak form of NMP [K43].

6.3 About The Physical Interpretation Of The Parameters Characterizing Algebraic Extension Of Rationals In TGD Framework

It seems that Galois group is naturally associated with the hierarchy $h_{eff}/h = n$ of effective Planck constants defined by the hierarchy of quantum criticalities. n would naturally define the maximal order for the element of Galois group. The analog of singular covering with that of $z^{1/n}$ would suggest that Galois group is very closely related to the conformal symmetries and its action induces permutations of the sheets of the covering of space-time surface.

Without any additional assumptions the values of n and ramified primes are completely independent so that the conjecture that the magnetic flux tube connecting the wormhole contacts associated with elementary particles would not correspond to very large n having the p-adic prime p characterizing particle as factor ($p = M_{127} = 2^{127} - 1$ for electron). This would not induce any catastrophic changes.

TGD based physics could however change the situation and reduce number theoretical degrees of freedom: the intuitive hypothesis that p divides n might hold true after all.

1. The strong form of GCI implies strong form of holography. One implication is that the WCW Kähler metric can be expressed either in terms of Kähler function or as anti-commutators of super-symplectic Noether super-charges defining WCW gamma matrices. This realizes what can be seen as an analog of Ads/CFT correspondence. This duality is much more general. The following argument supports this view.

(a) Since fermions are localized at string world sheets having ends at partonic 2-surfaces, one expects that also Kähler action can be expressed as an effective stringy action. It is natural to assume that string area action is replaced with the area defined by the effective metric of string world sheet expressible as anti-commutators of Kähler-Dirac gamma matrices defined by contractions of canonical momentum currents with embedding space gamma matrices. It string tension is proportional to h_{eff}^2, string length scales as h_{eff}.

(b) AdS/CFT analogy inspires the view that strings connecting partonic 2-surfaces serve as correlates for the formation of - at least gravitational - bound states. The distances between string ends would be of the order of Planck length in string models and one can argue that gravitational bound states are not possible in string models and this is the basic reason why one has ended to landscape and multiverse non-sense.

2. In order to obtain reasonable sizes for astrophysical objects (that is sizes larger than Schwartschild radius $r_s = 2GM$) For $\hbar_{eff} = \hbar_{gr} = GMm/v_0$ one obtains reasonable sizes for astrophysical objects. Gravitation would mean quantum coherence in astrophysical length scales.

3. In elementary particle length scales the value of h_{eff} must be such that the geometric size of elementary particle identified as the Minkowski distance between the wormhole contacts defining the length of the magnetic flux tube is of order Compton length - that is p-adic length scale proportional to $\sqrt{p}$. Note that dark physics would be an essential element already at elementary particle level if one accepts this picture also in elementary particle mass scales. This requires more precise specification of what darkness in TGD sense really means.

   One must however distinguish between two options.

   (a) If one assumes $n \simeq \sqrt{p}$, one obtains a large contribution to classical string energy as $\Delta \sim m_{CP_2}^2 L_p/\hbar_{eff}^2 \sim m_{CP_2}/\sqrt{p}$, which is of order particle mass. Dark mass of this size looks un-feasible since p-adic mass calculations assign the mass with the ends wormhole contacts. One must be however very cautious since the interpretations can change.

   (b) Second option allows to understand why the minimal size scale associated with CD characterizing particle correspond to secondary p-adic length scale. The idea is that the string can be thought of as being obtained by a random walk so that the distance between its ends is proportional to the square root of the actual length of the string in the induced metric. This would give that the actual length of string is proportional to p and n is also proportional to p and defines minimal size scale of the CD associated with the particle. The dark contribution to the particle mass would be $\Delta m \sim m_{CP_2}^2 L_p/\hbar_{eff}^2 \sim m_{CP_2}/p$, and completely negligible suggesting that it is not easy to make the dark side of elementary visible.

4. If the latter interpretation is correct, elementary particles would have huge number of hidden degrees of freedom assignable to their CDs. For instance, electron would have $n = 2^{127} - 1 \simeq 10^{38}$ hidden discrete degrees of freedom and would be rather intelligent system - 127 bits is the estimate- and thus far from a point-like idiot of standard physics. Is it a mere accident that the secondary p-adic time scale of electron is .1 seconds - the fundamental biorhythm - and the size scale of the minimal CD is slightly large than the circumference of Earth?

   Note however, that the conservation option assuming that the magnetic flux tubes connecting the wormhole contacts representing elementary particle are in $h_{eff}/h = 1$ phase can be considered as conservative option.

7 p-Adicization and adelic physics

This section is devoted to the challenges related to p-adicization and adelization of physics in which the correspondence between real and p-adic numbers via canonical identification serves as the basic building brick. Also the problems associated with p-adic variants of integral, Fourier analysis, Hilbert space, and Riemann geometry should be solved in a way respecting fundamental symmetries and their p-adic variants must be met. The notion of number theoretical universality

(NTU) plays a key role here. One should also answer to questions about the origin of preferred primes and p-adic length scale hypothesis.

7.1 Challenges

The basic challenges encountered are construction of the p-adic variants of real number based physics, understanding their relationship to real physics, and the fusion of various physics to single coherent whole.

The p-adicization of real physics is not just a straightforward formal generalization of scattering amplitudes of existing theories but requires a deeper understanding of the physics involved. The interpretation of p-adic physics as correlate for cognition and imagination is an important guideline and will be discussed l in more detail in separate section.

Definite integral and Fourier analysis are basic elements of standard physics and their generalization to the p-adic context defines a highly non-trivial challenge. Also the p-adic variants of Riemann geometry and Hilbert space are suggestive. There are however problems.

1. There are problems associated with p-adic definite integral. Riemann sum does not make sense since it approaches zero if the p-adic norm of discretization unit approaches zero. The problems are basically due to the absence of well-orderedness essential for the definition of definite integral and differential forms and their integrals.

   Residue integration might make sense in finite angle resolution. For algebraic extension containing $e^{i\pi/n}$ the number theoretically universal approximation $i\pi = n(e^{i\pi/n} - 1)$ could be used. In twistor approach integrations reduce to multiple residue integrations and since twistor approach generalizes in TGD framework, this approach to integration is very attractive.

   Positivity is a central notion in twistor Grassmannian approach [B1]. Since canonical identification maps p-adic numbers to non-negative real numbers, there is a strong temptation to think that positivity relates to NTU [L6].

2. There are problems with Fourier analysis. The naïve generalization of trigonometric functions by replacing e^{ix} with its p-adic counterpart is not physical. Same applies to e^x. Algebraic extensions are needed to get roots of unity ad e as counterparts of the phases and discretization is necessary and has interpretation in terms of finite resolution for angle/phase and its hyperbolic counterpart.

3. The notion of Hilbert space is problematic. The naïve generalization of Hilbert space norm square $|x|^2 = \sum x_n \overline{x}_n$ for state $(x_1, x_2, ...)$ can vanish p-adically. Also here NTU could help. State would contain as coefficients only roots of e and unity and only the overall factor could be p-adic number. Coefficients could be restricted to the algebraic numbers generating the algebraic extension of rational numbers and would not contain powers of p or even ordinary p-adic numbers expect in the overall normalization factor.

Second challenge relates to the relationship between real and p-adic physics. Canonical identification (CI) $\sum x_n p^n \rightarrow \sum x_n p^{-n}$ or some of its variants should play an important role. CI is expected to map the invariants appearing in scattering amplitudes to their real counterparts.

1. Real and p-adic variants of space-time surfaces should exist and relate to each other somehow. Is this relationship local and involve CI at space-time level or embedding space level? Or is it only a global and non-local assignment of preferred real extremals to their p-adic counterparts? Or is between these extreme options and involves algebraic discretization of the space-time surface weakening the strong form of SH as already proposed? How do real and p-adic embedding spaces relate to each other and can this relationship induce local correspondence between preferred extremals (PEs) [K3, K6, K4]?

2. NTU in some sense is a highly suggestive approach to these questions and would suggest that canonical identification applies to isometry invariants whereas angles and hyperbolic angles, or rather the corresponding "phases" belonging to an extension of p-adics containing roots of e and roots of unity are mapped to themselves. Note that the roots of e define extensions of

rationals, which induce finite dimensional algebraic extensions of p-adic numbers. This would make possible to define embedding space in accordance with NTU. Also the Hilbert space could be defined by requiring that its points correspond to number theoretically universal angles expressible in terms of roots of unity.

3. What about real and p-adic variants of WCW? Are they needed at all? Or could their existence be used as a powerful constraint on real physics? The representability of WCW as a union of infinite-dimensional symmetric spaces labelled by zero modes suggests that the same description applies at the level of WCW and embedding space.

   One cannot circumvent the question about how to generalize functional integral from real WCW to p-adic WCWs. In particular, what is the p-adic variant of the action defining the dynamics of space-time surfaces. In the case of exponent of action the p-adic variant could be defined by assuming algebraic universality: again the roots of e and of unity would be in central role. Also the Kähler structure of WCW implying that Gaussian and metric determinants cancel each other in functional integral, would be absolutely crucial.

   One must remember that the exponents of action for scattering amplitudes for the stationary phase extremal cancel from the path integral representation of scattering amplitudes. Also now this mechanism would allow to get rid of the poorly defined exponent for single minimum. If there is sum over scattering amplitudes assignable to different maxima, normalization sould give ratios of these exponents for different extrema/maxima and only these ratios should belong to the extension of rationals.

   The zero modes of WCW metric are invariants of supersymplectic group so that canonical identification could relate their real and p-adic variants. Zero modes could break NTU and would be behind p-adic thermodynamics and dependence of mass scale on p-adic prime.

The third challenge relates to the fusion of p-adic physics and real physics to a larger structure. Here a generalization of number concept obtained by glueing reals and various p-adics together along an extension of rational numbers inducing the extensions of p-adic numbers is highly suggestive. Adeles associated with the extension of rationals are highly attractive and closely related notion. Real and various p-adic physics would be correlates for sensory and cognitive aspects of the same universal physics rather than separate physics in this framework. One important implication of this view is that real entropy and p-adic negentropies characterize the same entanglement with coefficients in an extension of rationals.

NTU for hyperbolic and ordinary phases is definetely the central idea. How the invariance of angles under conformal transformations does relate to this? Could one perhaps define a discretized version of conformal symmetry preserving the phases defined by the angles between vectors assignable with the tangent spaces of discretized geometric structures and thus respecting NTU? Of should one apply conformal symmetry at Lie algebra level only?

7.2 NTU and the correspondence between real and p-adic physics

p-Adic real correspondence is certainly the basic problem of p-adicization and adelization. One can make several general questions about p-adic real correspondence and canonical identification inspired by p-adic mass calculations.

How generally p-adic real correspondence does apply? Could canonical identification for group invariants combined with direct identification of ordinary and hyperbolic phases identified as roots of unity and e apply at WCW and embedding space level having maximally symmetric geometries? Could this make sense even at space-time level as a correspondence induced from embedding space level [L10]? Does canonical identification apply locally for the discretizations of space-time surface or only globally for the parameters characterizing PEs (string world sheets and partonic 2-surfaces by SH), which are general coordinate invariant and Poincare invariant quantities?

The following vision seems to be the most feasible one found hitherto.

1. Preservation of symmetries and continuity compete. Lorenz transformations do not commute with canonical identification. This suggests that canonical identification applies only to Lorentz invariants formed from quantum numbers. This is enough in the case of scattering amplitudes. Canonical identification applies only to isometry invariants at the level

of WCW and the phases/exponents of ordinary/hyperbolic angles correspond to numbers in the algebraic extension common to extensions of rationals and various p-adics.

2. Canonical identification applies at the level of momentum space and maps p-adic Lorentz invariants of scattering amplitudes to their real counterparts. Phases of angles and their hyperbolic counterparts should correspond to parameters defining extension and should be mapped as such to their p-adic counterparts.

3. The constraints coming from GCI and symmetries do not allow local correspondence but allow to consider its discretized version at space-time leve induced by the correspondence at the level of embedding space.

   This requires the restriction of isometries and other symmetries to algebraic subgroups defined by the extension of rationals. This would imply reduction of symmetry due to finite cognitive/measurement resolution and should be acceptable. If one wants to realize the ideas about imagination, discretization must be applied also for the space-time interior meaning partial breaking of SH and giving rise to dark matter degrees freedom in TGD sense. SH could apply in real sector for realizable imaginations only. Note that the number of algebraic points of space-time surface is expected to be relatively small.

The correspondence must be considered at the level of embedding space, space-time, and WCW.

1. At the level of embedding space p-adic–real correspondence is induced by points in extension of rationals and is totally discontinuous. This requires that space-time dimension is smaller than embedding space dimension.

2. At space-time level the correspondence involves field equations derivable from a local variational principle make sense also p-adically although the action itself is ill-defined as 4-D integral. The notion of p-adic PE makes sense by strong form of holography applied to 2-surfaces in the intersection. p-Adically however only the vanishing of Noether currents for a sub-algebra of the super-symplectic algebra might make sense. This condition is stronger than the vanishing of Noether charges defined by 3-D integrals.

3. Correspondence at the level of WCW can make sense and reduces to that for string world sheets and partonic 2-surfaces by SH. Real and p-adic 4-surfaces would be obtained by algebraic continuation as PEs from 2-surfaces by assuming that the space-time surface contains subset of points of embedding space belonging to the extension of rationals. p-Adic pseudo constants make p-adic continuation easy. Real continuation need not exist always. p-Adic WCW would be considerably larger than real WCW and make possible a predictive quantum theory of imagination and cognition.

   What I have called intersection of realities and p-adicities can be identified as the set of 2-surfaces plus algebraic discretization of space-time interior. Also the values of induced spinor fields at the points of discretization must be given. The parameters characterizing the extremals (say coefficients of polynomials) - WCW coordinates - would be in extension of rationals inducing a finite-D extension of p-adic number fields.

   The hierarchy of algebraic extensions induces an evolutionary hierarchy of adeles. The interpretation could be as a mathematical correlate for cosmic evolution realized at the level of the core of WCW defined by the intersection? 2-surfaces could be called space-time genes.

4. Also the p-adic variant Kähler action or at least the exponent of Kähler action defining vacuum functional should be obtainable by algebraic continuation. The weakest condition states that the ratios of action exponents for the maxima of Kähler function to the sum of action exponents for maxima belong to the extension. Without this condition the hopes of satisfying NTU seem rather meager.

7.3 NTU at space-time level

What about NTU at space-time level? NTU requires a correspondence between real and p-adic numbers and the details of this corresponds have been a long standing problem.

1. The recent view about the correspondence between real PEs to their p-adic counterparts does not demand discrete local correspondence assumed in the earlier proposal [K46]. The most abstract approach would give up the local correspondence at space-time level altogether, and restrict the preferred coordinates of WCW (having maximal group of isometries) to numbers in the extension of rationals considered. WCW would be discretized.

   Intuitively a more realistic view is a correspondence at space-time level in the sense that real and p-adic space-time sheets intersect at points belonging to the extension of rationals and defining "cognitive representations". Only some p-adic space-time surfaces would have real counterpart.

2. The strongest form of NTU would require that the allowed points of embedding space belonging an extension of rationals are mapped as such to corresponding extensions of p-adic number fields (no canonical identification). At embedding space level this correspondence would be extremely discontinuous. The "spines" of space-time surfaces would however contain only a subset of points of extension, and a natural resolution length scale could emerge and prevent the fluctuation. This could be also seen as a reason for why space-times surfaces must be 4-D. The fact that the curve $x^n + y^n = z^n$ has no rational points for $n > 2$, raises the hope that the resolution scale could emerge spontaneously.

3. The notion of monadic geometry discussed in detail in [L10] would realize this idea. Define first a number theoretic discretization of embedding space in terms of points, whose coordinates in group theoretically preferred coordinate system belong to the extension of rationals considered. One can say that these algebraic points are in the intersection of reality and various p-adicities. Overlapping open sets assigned with this discretization define in the real sector a covering by open sets. In p-adic sector compact-open-topology allows to assign with each point 8^{th} Cartesian power of algebraic extension of p-adic numbers. These compact open sets define analogs for the monads of Leibniz and p-adic variants of field equations make sense inside them.

   The monadic manifold structure of H is induced to space-time surfaces containing discrete subset of points in the algebraic discretization with field equations defining a continuation to space-time surface in given number field, and unique only in finite measurement resolution. This approach would resolve the tension between continuity and symmetries in p-adic–real correspondence: isometry groups would be replaced by their sub-groups with parameters in extension of rationals considered and acting in the intersection of reality and p-adicities.

   The Galois group of extension acts non-trivially on the "spines" of space-time surfaces. Hence the number theoretical symmetries act as physical symmetries and define the orbit of given space-time surface as a kind of covering space. The coverings assigned to the hierarchy of Planck constants would naturally correspond to Galois coverings and dark matter would represent number theoretical physics.

   This would give rise to a kind of algebraic hierarchy of adelic 4-surfaces identifiable as evolutionary hierarchy: the higher the dimension of the extension, the higher the evolutionary level.

7.4 NTU and WCW

7.4.1 p-Adic–real correspondence at the level of WCW

It has not been obvious whether one should perform p-adicization and adelization at the level of WCW. Minimalist could argue that scattering amplitudes are all we want and that their p-adicization and adelization by algebraic continuation can be tolerated only if it can give powerful enough constraints on the amplitudes.

1. The anti-commutations for fermionic oscillator operators are number theoretically universal. Supersymmetry suggests that also WCW bosonic degrees of freedom satisfy NTU. This could mean that the coordinates of p-adic WCW consist of super-symplectic invariants mappable by canonical identification to their real counterparts plus phases and their hyperbolic counterparts expressible as genuinely algebraic numbers common to all number fields. This kind of coordinates are naturally assignable to symmetric spaces [L10].

2. Kähler structure should be mapped from p-adic to real sector and vice versa. Vacuum functional identified as exponent of action should be NTU. Algebraic continuation defined by SH involves p-adic pseudo constants. All p-adic continuations by SH should correspond to the same value of exponent of action obtained by algebraic continuation from its real value. The degeneracy associated with p-adic pseudo-constants would be analogous to gauge invariance - imagination in TGD inspired theory of consciousness.

3. Ist it possible have NTU for WCW functional integration? Or is it enough to realize NTU for scattering amplitudes only. What seems clear that functional integral must reduce to a discrete sum. Physical intuition suggests a sum over maxima of Kähler function forming a subset of PEs representing stationary points. One cannot even exclude the possibility that the set of PEs is discrete and that one can sum over all of them.

   Restriction to maximum/stationary phase approximation gives rise to sum over exponents multiplied with Gaussian determinants. The determinant of Kähler metric however cancels the Gaussian determinants, and one obtains only a sum over the exponents of action.

   The breaking of strong NTU could happen: consider only p-adic mass calculations. This breaking is however associated with the parts of quantum states assignable to the boundaries of CD, not with the vacuum functional.

7.4.2 NTU for functional integral

Number theoretical vision relies on NTU. In fermionic sector NTU is necessary: one cannot speak about real and p-adic fermions as separate entities and fermionic anti-commutation relations are indeed number theoretically universal.

What about NTU in case of functional integral? There are two opposite views.

1. One can define p-adic variants of field equations without difficulties if preferred extremals are minimal surface extremals of Kähler action so that coupling constants do not appear in the solutions. If the extremal property is determined solely by the analyticity properties as it is for various conjectures, it makes sense independent of number field. Therefore there would be no need to continue the functional integral to p-adic sectors. This in accordance with the philosophy that thought cannot be put in scale. This would be also the option favored by pragmatist.

2. Consciousness theorist might argue that also cognition and imagination allow quantum description. The supersymmetry NTU should apply also to functional integral over WCW (more precisely, its sector defined by CD) involved with the definition of scattering amplitudes.

1. Key observations

The general vision involves some crucial observations.

1. Only the expressions for the scatterings amplitudes should should satisfy NTU. This does not require that the functional integral satisfies NTU.

2. Since the Gaussian and metric determinants cancel in WCW Kähler metric the contributions form maxima are proportional to action exponentials $exp(S_k)$ divided by the $\sum_k exp(S_k)$. Loops vanish by quantum criticality.

3. Scattering amplitudes can be defined as sums over the contributions from the maxima, which would have also stationary phase by the double extremal property made possible by the complex value of α_K. These contributions are normalized by the vacuum amplitude.

   It is enough to require NTU for $X_i = exp(S_i)/\sum_k exp(S_k)$. This requires that $S_k - S_l$ has form $q_1 + q_2 i\pi + q_3 log(n)$. The condition brings in mind homology theory without boundary operation defined by the difference $S_k - S_l$. NTU for both S_k and $exp(S_k)$ would only values of general form $S_k = q_1 + q_2 i\pi + q_3 log(n)$ for S_k and this looks quite too strong a condition.

4. If it is possible to express the 4-D exponentials as single 2-D exponential associated with union of string world sheets, vacuum functional disappears completely from consideration! There is only a sum over discretization with the same effective action and one obtains purely combinatorial expression.

2. What does one mean with functional integral?

The definition of functional integral in WCW is one of the key technical problems of quantum TGD [K43]. NTU states that the integral should be defined simultaneously in all number fields in the intersection of real and p-adic worlds defined by string world sheets and partonic 2-surfaces with WCW coordinates in algebraic extension of rationals and allowing by strong holography continuation to 4-D space-time surface. NTU is powerful constraint and could help in this respect.

1. Path integral is not in question. Rather, the functional integral is analogous to Wiener integral and perhaps allows identification as a genuine integral in the real sector. In p-adic sectors algebraic continuation should give the integral and here number theoretical universality gives excellent hopes. The integral would have exactly the same form in real and p-adic sector and expressible solely in terms of algebraic numbers characterizing algebraic extension and finite roots of e and roots of unity $U_n = exp(i2\pi/n)$ in algebraic extension of p-adic numbers.

   Since vacuum functional $exp(S)$ is exponential of complex action S, the natural idea is that only rational powers e^q and roots of unity and phases $exp(i2\pi q)$ are involved and there is no dependence on p-adic prime p! This is true in the integer part of q is smaller than p so that one does not obtain e^{kp}, which is ordinary p-adic number and would spoil the number theoretic universality. This condition is not possible to satisfy for all values of p unless the value of Kähler function is smaller than 2. One might consider the possibility that the allow primes are above some minimum value.

   The minimal solution to NTU conditions is that the ratios of action exponentials for maxima of Kähler function to the sum of these exponentials belong to the extension of rationals considered.

2. What does one mean with functional integral? TGD is expected to be an integrable in some sense. In integrable QFTs functional integral reduces to a sum over stationary points of the action: typically only single point contributes - at least in good approximation.

   For real α_K and Λ vacuum functional decomposes to a product of exponents of real contribution from Euclidian regions ($\sqrt{g_4}$ real) and imaginary contribution Minkowskian regions ($\sqrt{g_4}$ imaginary). There would be no exchange of momentum between Minkowskian and Euclidian regions. For complex values of α_K [K13] situation changes and Kähler function as real part of action receives contributions from both Euclidian and Minkowskian regions. The imaginary part of action has interpretation as analog of Morse function and action as it appears in QFTs. Now saddle points must be considered.

   PEs satisfy extremely strong conditions [K3, K4]. All classical Noether charges for a sub-algebra associated with super-symplectic algebra and isomorphic to the algebra itself vanish at both ends of CD. The conformal weights of this algebra are $n > 0$-ples of those for the entire algebra. What is fascinating that the condition that the preferred extremals are minimal surface extremals of Kähler action could solve these conditions and guarantee also NTU at the level of space-time surfaces. Supersymplectic boundary conditions at the ends of CD would however pose number theoretic conditions on the coupling parameters. In p-adic case the conditions should reduce to purely local conditions since p-adic charges are not well-defined as integrals.

3. In TGD framework one is constructing zero energy states rather calculating the matrix elements of S-matrix in terms of path integral. This gives certain liberties but a natural expectation is that functional integral as a formal tool at least is involved.

   Could one *define* the functional integral as a discrete sum of contributions of standard form for the preferred extremals, which correspond to maxima in Euclidian regions and associated stationary phase points in Minkowskian regions? Could one assume that WCW spinor field is concentrated along single maximum/stationary point.

Quantum classical correspondence suggests that in Cartan algebra isometry charges are equal to the quantal charges for quantum states expressible in number theoretically universal manner in terms of fermionic oscillator operators or WCW gamma matrices? Even stronger condition would be that classical correlation functions are identical with quantal ones for allowed space-time surfaces in the quantum superposition. Could the reduction to a discrete sum be interpreted in terms of a finite measurement resolution?

4. In QFT Gaussian determinants produce problems because they are often poorly defined. In the recent case they could also spoil the NTU based on the exceptional properties of e. In the recent case however Gaussian determinant and metric determinant for Kähler metric cancel each other and this problem disappears. One could obtain just a sum over products of roots of e and roots of unity. Thus also Kähler structure seems to be crucial for the dream about NTU.

7.5 Breaking of NTU at the level of scattering amplitudes

NTU in strong sense could be broken at the level of scattering amplitudes. At space-time level the breaking does not look natural in the recent framework. Consider only p-adic mass calculations predicting that mass scale depends on p-adic prime. Also for the action strong form of NTU might fail for small p-adic primes since the value of the real part of action would be larger than than p. Should one allow this? What does one actually mean with NTU in the case of action?

Canonical identification is an important element of p-adic mass calculations and might also be needed to map p-adic variants of scattering amplitudes to their real counterparts. The breaking of NTU would take place, when the canonical real valued image of the p-adic scattering amplitude differs from the real scattering amplitude. The interpretation would be in terms of finite measurement resolution. By the finite measurement/cognitive resolution characterized by p one cannot detect the difference.

The simplest form of the canonical identification is $x = \sum_n x_n p^n \rightarrow \sum x_n p^{-n}$. Product xy and sum $x + y$ do not in general map to product and sum in canonical identification. The interpretation would be in terms of a finite measurement resolution: $(xy)_R = x_R y_R$ and $(x+y)_R = x_R + y_R$ only modulo finite measurement resolution. p-Adic scattering amplitudes are obtained by algebraic continuation from the intersection by replacing algebraic number valued parameters (such as momenta) by general p-adic numbers. The real images of these amplitudes under canonical identification are in general not identical with real scattering amplitudes the interpretation being in terms of a finite measurement resolution.

In p-adic thermodynamics NTU in the strong sense fails since thermal masses depend on p-adic mass scale. NTU can be broken by the fermionic matrix elements in the functional integral so that the real scattering amplitudes differ from the canonical images of the p-adic scattering amplitudes. For instance, the elementary particle vacuum functionals in the space of Teichmueller parameters for the partonic 2-surfaces and string world sheets should break NTU [K9].

7.6 NTU and the spectrum of Kähler coupling strength

During years I have made several attempts to understand coupling evolution in TGD framework. The most convincing proposal has emerged rather recently and relates the spectrum of $1/\alpha_K$ to that for the zeros of Riemann zeta [K13] and to the evolution of the electroweak U(1) couplings strength.

1. The first idea dates back to the discovery of WCW Kähler geometry defined by Kähler function defined by Kähler action (this happened around 1990) [K18]. The only free parameter of the theory is Kähler coupling strength α_K analogous to temperature parameter α_K postulated to be is analogous to critical temperature. Whether only single value or entire spectrum of of values α_K is possible, remained an open question.

   About decade ago I realized that Kähler action is *complex* receiving a real contribution from space-time regions of Euclidian signature of metric and imaginary contribution from the Minkoswkian regions. Euclidian region would give Kähler function and Minkowskian regions analog of QFT action of path integral approach defining also Morse function. Zero energy ontology (ZEO) [K47] led to the interpretation of quantum TGD as complex square root of thermodynamics so that the vacuum functional as exponent of Kähler action could be identified as a complex square root of the ordinary partition function. Kähler function would correspond to the real contribution Kähler action from Euclidian space-time regions. This led to ask whether also Kähler coupling strength might be complex: in analogy with the complexification of gauge coupling strength in theories allowing magnetic monopoles. Complex α_K could allow to explain CP breaking. I proposed that instanton term also reducing to Chern-Simons term could be behind CP breaking.

The problem is that the dynamics in Minkowskian and Euclidian regions decouple completely and if Euclidian regions serve as space-time correlates for physical objects, there would be no exchanges of classical charges between physical objects. Should one conclude that α_K must be complex?

2. p-Adic mass calculations for 2 decades ago [K22] inspired the idea that length scale evolution is discretized so that the real version of p-adic coupling constant would have discrete set of values labelled by p-adic primes. The simple working hypothesis was that Kähler coupling strength is renormalization group (RG) invariant and only the weak and color coupling strengths depend on the p-adic length scale. The alternative ad hoc hypothesis considered was that gravitational constant is RG invariant. I made several number theoretically motivated ad hoc guesses about coupling constant evolution, in particular a guess for the formula for gravitational coupling in terms of Kähler coupling strength, action for CP_2 type vacuum extremal, p-adic length scale as dimensional quantity [K38]. Needless to say these attempts were premature and a hoc.

3. The vision about hierarchy of Planck constants $h_{eff} = n \times h$ and the connection $h_{eff} = h_{gr} = GMm/v_0$, where $v_0 < c = 1$ has dimensions of velocity [?] forced to consider very seriously the hypothesis that Kähler coupling strength has a spectrum of values in one-one correspondence with p-adic length scales. A separate coupling constant evolution associated with h_{eff} induced by $\alpha_K \propto 1/\hbar_{eff} \propto 1/n$ looks natural and was motivated by the idea that Nature is theoretician friendly: when the situation becomes non-perturbative, Mother Nature comes in rescue and an h_{eff} increasing phase transition makes the situation perturbative again.

   Quite recently the number theoretic interpretation of coupling constant evolution [K43] [L5] in terms of a hierarchy of algebraic extensions of rational numbers inducing those of p-adic number fields encouraged to think that $1/\alpha_K$ has spectrum labelled by primes and values of h_{eff}. Two coupling constant evolutions suggest themselves: they could be assigned to length scales and angles which are in p-adic sectors necessarily discretized and describable using only algebraic extensions involve roots of unity replacing angles with discrete phases.

4. Few years ago the relationship of TGD and GRT was finally understood [K40]. GRT space-time is obtained as an approximation as the sheets of the many-sheeted space-time of TGD are replaced with single region of space-time. The gravitational and gauge potential of sheets add together so that linear superposition corresponds to set theoretic union geometrically. This forced to consider the possibility that gauge coupling evolution takes place only at the level of the QFT approximation and α_K has only single value. This is nice but if true, one does not have much to say about the evolution of gauge coupling strengths.

5. The analogy of Riemann zeta function with the partition function of complex square root of thermodynamics suggests that the zeros of zeta have interpretation as inverses of complex temperatures $s = 1/\beta$. Also $1/\alpha_K$ is analogous to temperature. This led to a radical idea to be discussed in detail in the sequel.

   Could the spectrum of $1/\alpha_K$ reduce to that for the zeros of Riemann zeta or - more plausibly - to the spectrum of poles of fermionic zeta $\zeta_F(ks) = \zeta(ks)/\zeta(2ks)$ giving for $k = 1/2$ poles as zeros of zeta and as point $s = 2$? ζ_F is motivated by the fact that fermions are the only fundamental particles in TGD and by the fact that poles of the partition function are naturally associated with quantum criticality whereas the vanishing of ζ and varying sign allow no natural physical interpretation.

The poles of $\zeta_F(s/2)$ define the spectrum of $1/\alpha_K$ and correspond to zeros of $\zeta(s)$ and to the pole of $\zeta(s/2)$ at $s=2$. The trivial poles for $s=2n$, $n=1,2,..$ correspond naturally to the values of $1/\alpha_K$ for different values of $h_{eff}=n\times h$ with n even integer. Complex poles would correspond to ordinary QFT coupling constant evolution. The zeros of zeta in increasing order would correspond to p-adic primes in increasing order and UV limit to smallest value of poles at critical line. One can distinguish the pole $s=2$ as extreme UV limit at which QFT approximation fails totally. CP_2 length scale indeed corresponds to GUT scale.

6. One can test this hypothesis. $1/\alpha_K$ corresponds to the electroweak U(1) coupling strength so that the identification $1/\alpha_K = 1/\alpha_{U(1)}$ makes sense. One also knows a lot about the evolutions of $1/\alpha_{U(1)}$ and of electromagnetic coupling strength $1/\alpha_{em} = 1/[cos^2(\theta_W)\alpha_{U(1)}$. What does this predict?

   It turns out that at p-adic length scale $k=131$ ($p\simeq 2^k$ by p-adic length scale hypothesis, which now can be understood number theoretically [K43]) fine structure constant is predicted with .7 per cent accuracy if Weinberg angle is assumed to have its value at atomic scale! It is difficult to believe that this could be a mere accident because also the prediction evolution of $\alpha_{U(1)}$ is correct qualitatively. Note however that for $k=127$ labelling electron one can reproduce fine structure constant with Weinberg angle deviating about 10 per cent from the measured value of Weinberg angle. Both models will be considered.

7. What about the evolution of weak, color and gravitational coupling strengths? Quantum criticality suggests that the evolution of these couplings strengths is universal and independent of the details of the dynamics. Since one must be able to compare various evolutions and combine them together, the only possibility seems to be that the spectra of gauge coupling strengths are given by the poles of $\zeta_F(w)$ but with argument $w=w(s)$ obtained by a global conformal transformation of upper half plane - that is Möbius transformation (see `http://tinyurl.com/gwjs85b`) with real coefficients (element of $GL(2,R)$) so that one as $\zeta_F((as+b)/(cs+d))$. Rather general arguments force it to be and element of $GL(2,Q)$, $GL(2,Z)$ or maybe even $SL(2,Z)$ ($ad-bc=1$) satisfying additional constraints. Since TGD predicts several scaled variants of weak and color interactions, these copies could be perhaps parameterized by some elements of $SL(2,Z)$ and by a scaling factor K.

   Could one understand the general qualitative features of color and weak coupling contant evolutions from the properties of corresponding Möbius transformation? At the critical line there can be no poles or zeros but could asymptotic freedom be assigned with a pole of $cs+d$ and color confinement with the zero of $as+b$ at real axes? Pole makes sense only if Kähler action for the preferred extremal vanishes. Vanishing can occur and does so for massless extremals characterizing conformally invariant phase. For zero of $as+b$ vacuum function would be equal to one unless Kähler action is allowed to be infinite: does this make sense?. One can however hope that the values of parameters allow to distinguish between weak and color interactions. It is certainly possible to get an idea about the values of the parameters of the transformation and one ends up with a general model predicting the entire electroweak coupling constant evolution successfully.

To sum up, the big idea is the identification of the spectra of coupling constant strengths as poles of $\zeta_F((as+b/)(cs+d))$ identified as a complex square root of partition function with motivation coming from ZEO, quantum criticality, and super-conformal symmetry; the discretization of the RG flow made possible by the p-adic length scale hypothesis $p\simeq k^k$, k prime; and the assignment of complex zeros of ζ with p-adic primes in increasing order. These assumptions reduce the coupling constant evolution to four real rational or integer valued parameters (a,b,c,d). In the sequel this vision is discussed in more detail.

7.7 Generalization of Riemann zeta to Dedekind zeta and adelic physics

7.8 Generalization of Riemann zeta to Dedekind zeta and adelic physics

A further insight to adelic physics comes from the possible physical interpretation of the L-functions appearing also in Langlands program [K20]. The most important L-function would be generalization of Riemann zeta to extension of rationals. I have proposed several roles for ζ, which would be the simplest L-function assignable to rational primes, and for its zeros.

1. Riemann zeta itself could be identifiable as an analog of partition function for a system with energies given by logarithms of prime. One can define also the fermionic counterpart of ζ as ζ_F. In ZEO this function could be regarded as complex square root of thermodynamical partition function in accordance with the interpretation of quantum theory as complex square root of thermodynamics.

2. The zeros of zeta could define the conformal weights for the generators of super-symplectic algebra so that the number of generators would be infinite. The rough idea - certainly not correct as such except at the limit of infinitely large CD - is that the scaling operator $L_0 = r_M d/dr_M$, where r_M is light-like coordinate of light-cone boundary (containing upper or lower boundary of the causal diamond (CD)), has as eigenfunctions the functions $(r_M/r_0)^{s_n}$ $s_n = 1/2 + iy_n$, where s_n is the radial conformal weight identified as complex zero of ζ. Periodic boundary conditions for CD do not allow all possible zeros as conformal weights so that for given CD only finite subset corresponds to generators of the supersymplectic algebra. Conformal confinement would hold true in the sense that the sum $\sum_n s_n$ for physical states would be integer. Roots and their conjugates should appear as pairs in physical states.

3. On basis of numerical evidence Dyson [A10] (`http://tinyurl.com/hjbfsuv`) has conjectured that the Fourier transform for the set formed by zeros of zeta consists of primes so that one could regard zeros as one-dimensional quasi-crystal. This hypothesis makes sense if the zeros of zeta decompose into disjoint sets such that each set corresponds to its own prime (and its powers) and one has $p^{iy} = U_{m/n} = exp(i2\pi m/n)$ (see the appendix of [L5]). This hypothesis is also motivated by number theoretical universality [K43, K1].

4. I have considered the possibility [K13] that the discrete values for the inverse of the electroweak $U(1)$ coupling constant for a gauge field assignable to the Kähler form of CP_2 assignable to p-adic coupling constant evolution corresponds to poles of the fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ coming from $s_n/2$ (denominator) and pole at $s = 1$ (numerator) zeros of zeta assignable to rational primes. Note that also odd negative integers at real axis would be poles.

   It is also possible to consider scaling of the argument of $\zeta_F(s)$. More general coupling constant evolutions could correspond to $\zeta_F(m(s))$, where $m(s) = (as+b)/(cs+d)$ is Möbius transformation performed for the argument mapping upper complex plane to itself so that a, b, c, d are real and also rational by number theoretical universality.

Suppose for a moment that more precise formulations of these physics inspired conjectures make sense and even that their generalization for extensions K/Q of rationals holds true. This would solve a big part of adelic physics! Not surprisingly, the generalization of zeta function was proposed already by Dedekind (see `http://tinyurl.com/yarwbo6h`).

1. The definition of Dedekind zeta function ζ_K relies on the product representation and analytic continuation allows to deduce ζ_K elsewhere. One has a product over prime ideals of K/Q of rationals with the factors $1/(1-p^{-s})$ associated with the ordinary primes in Riemann zeta replaced with the factors $X(P) = 1/(1 - N_{K/Q}(P)^{-s})$, where P is prime for the integers $O(K)$ of extension and $N_{K/Q}(P)$ is the norm of P in the extension. In the region $s > 1$ where the product converges, ζ_K is non-vanishing and $s = 1$ is a pole of ζ_K. The functional identifies of ζ hold true for ζ_K as well. Riemann hypothesis is generalized for ζ_K.

2. It is possible to understand ζ_K in terms of a physical picture. By the results of `http://tinyurl.com/yckfjgpk` one has $N_{K/Q}(P) = p^r$, $r > 0$ integer. This implies that one can

arrange in ζ_K all primes P for which the norm is power or given p in the same group. The prime ideals p of ordinary integers decompose to products of prime ideals P of the extension: one has $p = \prod_{r=1}^{g} P_r^{e_r}$, where e_r is so called ramification index. One can say that each factor of ζ decomposes to a product of factors associated with corresponding primes P with norm power of p. In the language of physics, the particle state represented by p decomposes in improved resolution to a product of many-particle states consisting of e_r particles in state P_r, very much like hadron decomposes to quarks.

The norms of $N_{K/Q}(P_r) = p^{d_r}$ satisfy the condition $\sum_{r=1}^{g} d_r e_r = n$. Mathematician would say that the prime ideals of Q modulo p decompose in n-dimensional extension K to products of prime power ideals $P_r^{e_r}$ and that P_r corresponds to a finite field $G(p, d_r)$ with algebraic dimension d_r. The formula $\sum_{r=1}^{g} d_r e_r = n$ reflects the fact the dimension n of extension is same independent of p even when one has $g < n$ and ramification occurs.

Physicist would say that the number of degrees of freedom is n and is preserved although one has only $g < n$ different particle types with e_r particles having d_r internal degrees of freedom. The factor replacing $1/(1-p^{-s})$ for the general prime p is given by $\prod_{r=1}^{g} 1/(1-p^{-e_r d_r s})$.

3. There are only finite number of ramified primes p having $e_r > 1$ for some r and they correspond to primes dividing the so called discriminant D of the irreducible polynomial P defining the extension. $D \ mod \ p$ obviously vanishes if D is divisible by p. For second order polynomials $P = x^2 + bx + c$ equals to the familiar $D = b^2 - 4c$ and in this case the two roots indeed co-incide. For quadratic extensions with $D = b^2 - 4c > 0$ the ramified primes divide D.

   Remark: Resultant $R(P, Q)$ and discriminant $D(P) = R(P, dP/dx)$ are elegant tools used by number theorists to study extensions of rationals defined by irreducible polynomials (see `http://tinyurl.com/oyumsnk` and `http://tinyurl.com/p67rdgb`). From Wikipedia articles one finds an elegant proof for the facts that $R(P, Q)$ is proportional to the product of differences of the roots of P and Q, and D to the product of squares for the differences of distinct roots. $R(P, Q) = 0$ tells that two polynomials have a common root. $D \ mod \ p = 0$ tells that polynomial and its derivative have a common root so that there is a degenerate root modulo p and the prime is indeed ramified. For modulo p reduction of P the vanishing of $D(P) \ mod \ p$ follows if D is divisible by p. There exists clearly only a finite number of primes of this kind.

   Most primes are unramified and one has maximum number n of factors in the decomposition and $e_r = 1$: maximum splitting of p occurs. The factor $1/(1-p^{-s})$ is replaced with its n:th power $1/(1-p^{-s})^n$. The geometric interpretation is that space-time sheet is replaced with n-fold covering and each sheet gives one factor in the power. It is also possible to have a situation in which no splitting occurs and one as $e_r = 1$ for one prime $P_r = p$. The factor is in this case equal to $1/(1-p^{-ns})$.

From Wikipedia (see `http://tinyurl.com/yckfjgpk`) one learns that for Galois extensions L/K the ratio ζ_L/ζ_K is so called Artin L-function of the regular representation (group algebra) of Galois group factorizing in terms of irreps of $Gal(L/K)$ is *holomorphic* (no poles!) so that ζ_L must have also the zeros of ζ_K. This holds in the special case $K = Q$. Therefore extension of rationals can only bring new zeros but no new poles!

1. This result is quite far reaching if one accepts the hypothesis about super-symplectic conformal weights as zeros of ζ_K and the conjecture about coupling constant evolution. In the case of $\zeta_{F,K}$ this means new poles meaning new conformal weights due to increased complexity and a modification of the conjecture for the coupling constant evolution due to new primes in extension. The outcome looks physically sensible.

2. Quadratic field $Q(\sqrt{m})$ serves as example. Quite generally, the factorization of rational primes to the primes of extension corresponds to the factorization of the minimal polynomial for the generating element θ for the integers of extension and one has $p = P_i^{e_i}$, where e_i is ramification index. The norm of p factorizes to the produce of norms of $P_i^{e_i}$.

   Rational prime can either remain prime in which case $x^2 - m$ does not factorize mod p, split when $x^2 - m$ factorizes mod P, or ramify when it divides the discriminant of $x^2 - m =$

$4m$. From this it is clear that for unramfied primes the factors in ζ are replaced by either $1/(1-p^{-s})^2$ or $1/(1-p^{-2s}) = 1/(1-p^{-s})(1+p^{-s})$. For a finite number of unramified primes one can have something different.

For Gaussian primes with $m = -1$ one has $e_r = 1$ for $p\ mod\ 4 = 3$ and $e_r = 2$ for $p =\ mod\ 4 = 1$. z_K therefore decomposes into two factors corresponding to primes $p\ mod\ 4 = 3$ and $p\ mod\ 4 = 3$. One can extract out Riemann zeta and the remaining factor

$$\prod_{p\ mod\ 4=3} \frac{1}{(1-p^{-s})} \times \prod_{p\ mod\ 4=1} \frac{1}{(1+p^{-s})}$$

should be holomorphic and without poles but having possibly additional zeros at critical line. That ζ_K should possess also the poles of ζ as poles looks therefore highly non-trivial.

7.9 Other applications of NTU

NTU in the strongest form says that all numbers involved at "basic level" (whatever this means!) of adelic TGD are products of roots of unity and of power of a root of e. This is extremely powerful physics inspired conjecture with a wide range of possible mathematical applications.

1. For instance, vacuum functional defined as an exponent of action for preferred externals would be number of this kind. One could define functional integral as adelic operation in all number fields: essentially as sum of exponents of action for stationary preferred extremals since Gaussian and metric determinants potentially spoiling NTU would cancel each other leaving only the exponent.

2. The implications of NTU for the zeros of Riemann zeta [L5] will be discussed in more detail in the Appendix. Suffice it to say that the observations about Fourier transform for the distribution of loci of non-trivial zeros of zeta together with NTU leads to explicit proposal for the algebraic for of zeros of zeta. The testable proposal is that zeros decompose to disjoint classes $C(p)$ labelled by primes p and the condition that p^{iy} is root of unity in given class $C(p)$.

3. NTU generalises to all Lie groups. Exponents $exp(in_iJ_i/n)$ of lie-algebra generators define generalisations of number theoretically universal group elements and generate a discrete subgroup of compact Lie group. Also hyperbolic "phases" based on the roots $e^{m/n}$ are possible and make possible discretized NTU versions of all Lie-groups expected to play a key role in adelization of TGD.

   NTU generalises also to quaternions and octonions and allows to define them as number theoretically universal entities. Note that ordinary p-adic variants of quaternions and octonions do not give rise to a number field: inverse of quaternion can have vanishing p-adic variant of norm squared satisfying $\sum_n x_n^2 = 0$.

   NTU allows to define also the notion of Hilbert space as an adelic notion. The exponents of angles characterising unit vector of Hilbert space would correspond to roots of unity.

7.10 Going to the roots of p-adicity

The basic questions raised by the p-adic mass calculations concern the origin of preferred p-adic primes and of p-adic length scale hypothesis. One can also ask whether there might be a natural origin for p-adicity at the level of WCW.

7.10.1 Preferred primes as ramified primes for extensions of rationals?

7.10.2 Preferred primes as ramified primes for extensions of rationals?

The intuitive feeling is that the notion of preferred prime is something extremely deep and to me the deepest thing I know is number theory. Does one end up with preferred primes in number theory? This question brought to my mind the notion of *ramification of primes* (`http://tinyurl.`

com/hddljlf) (more precisely, of prime ideals of number field in its extension), which happens only for special primes in a given extension of number field, say rationals. Ramification is completely analogous to the degeneracy of some roots of polynomial and corresponds to criticality if the polynomial corresponds to criticality (catastrophe theory of Thom is one application). Could this be the mechanism assigning preferred prime(s) to a given elementary system, such as elementary particle? I have not considered their role earlier also their hierarchy is highly relevant in the number theoretical vision about TGD.

1. Stating it very roughly (I hope that mathematicians tolerate this sloppy language of physicist): as one goes from number field K, say rationals Q, to its algebraic extension L, the original prime ideals in the so called *integral closure* (http://tinyurl.com/js6fpvr) over integers of K decompose to products of prime ideals of L (prime ideal is a more rigorous manner to express primeness). Note that the general ideal is analog of integer.

   Integral closure for integers of number field K is defined as the set of elements of K, which are roots of some monic polynomial with coefficients, which are integers of K having the form $x^n + a_{n-1}x^{n-1} + ... + a_0$. The integral closures of both K and L are considered. For instance, integral closure of algebraic extension of K over K is the extension itself. The integral closure of complex numbers over ordinary integers is the set of algebraic numbers.

   Prime ideals of K can be decomposed to products of prime ideals of L: $P = \prod P_i^{e_i}$, where e_i is the ramification index. If $e_i > 1$ is true for some i, *ramification* occurs. P_i:s in question are like co-inciding roots of polynomial, which for in thermodynamics and Thom's catastrophe theory corresponds to criticality. Ramification could therefore be a natural aspect of quantum criticality and ramified primes P are good candidates for preferred primes for a given extension of rationals. Note that the ramification make sense also for extensions of given extension of rationals.

2. A physical analogy for the decomposition of ideals to ideals of extension is provided by decomposition of hadrons to valence quarks. Elementary particles becomes composite of more elementary particles in the extension. The decomposition to these more elementary primes is of form $P = \prod P_i^{e(i)}$, the physical analog would be the number of elementary particles of type i in the state (http://tinyurl.com/h9528pl). Unramified prime P would be analogous a state with e fermions. Maximally ramified prime would be analogous to Bose-Einstein condensate of e bosons. General ramified prime would be analogous to an e-particle state containing both fermions and condensed bosons. This is of course just a formal analogy.

3. There are two further basic notions related to ramification and characterizing it. *Relative discriminant* is the ideal divided by all ramified ideals in K (integer of K having no ramified prime factors) and relative different for P is the ideal of L divided by all ramified P_i:s (product of prime factors of P in L). These ideals represent the analogs of product of preferred primes P of K and primes P_i of L dividing them. These two integers ideals would characterize the ramification.

In TGD framework the extensions of rationals (http://tinyurl.com/h9528pl) and p-adic number fields (http://tinyurl.com/zq22tvb) are unavoidable and interpreted as an evolutionary hierarchy physically and cosmological evolution would gradually proceed to more and more complex extensions. One can say that string world sheets and partonic 2-surfaces with parameters of defining functions in increasingly complex extensions of prime emerge during evolution. Therefore ramifications and the preferred primes defined by them are unavoidable. For p-adic number fields the number of extensions is much smaller for instance for $p > 2$ there are only 3 quadratic extensions.

How could ramification relate to p-adic and adelic physics and could it explain preferred primes?

1. Ramified p-adic prime $P = P_i^e$ would be replaced with its e:th root P_i in p-adicization. Same would apply to general ramified primes. Each un-ramified prime of K is replaced with $e = K : L$ primes of L and ramified primes P with $\#\{P_i\} < e$ primes of L: the increase of algebraic dimension is smaller. An interesting question relates to p-adic length scale. What happens to p-adic length scales. Is p-adic prime effectively replaced with e:th root of p-adic

prime: $L_p \propto p^{1/2}L_1 \to p^{1/2e}L_1$? The only physical option is that the p-adic temperature for P would be scaled down $T_p = 1/n \to 1/ne$ for its e:th root (for fermions serving as fundamental particles in TGD one actually has $T_p = 1$). Could the lower temperature state be more stable and select the preferred primes as maximimally ramified ones? What about general ramified primes?

2. This need not be the whole story. Some algebraic extensions would be more favored than others and p-adic view about realizable imaginations could be involved. p-Adic pseudo constants are expected to allow p-adic continuations of string world sheets and partonic 2-surfaces to 4-D preferred extremals with number theoretic discretization. For real continuations the situation is more difficult. For preferred extensions - and therefore for corresponding ramified primes - the number of real continuations - realizable imaginations - would be especially large.

   The challenge would be to understand why primes near powers of 2 and possibly also of other small primes would be favored. Why for them the number of realizable imaginations would be especially large so that they would be winners in number theoretical fight for survival?

Can one make this picture more concrete? What kind of algebraic extensions could be considered?

1. In p-adic context a proper definition of counterparts of angle variables as phases allowing definition of the analogs of trigonometric functions requires the introduction of algebraic extension giving rise to some roots of unity. Their number depends on the angular resolution. These roots allow to define the counterparts of ordinary trigonometric functions - the naïve generalization based on Taylors series is not periodic - and also allows to defined the counterpart of definite integral in these degrees of freedom as discrete Fourier analysis. For the simplest algebraic extensions defined by $x^n - 1$ for which Galois group is abelian are are unramified so that something else is needed. One has decomposition $P = \prod P_i^{e(i)}$, $e(i) = 1$, analogous to n-fermion state so that simplest cyclic extension does not give rise to a ramification and there are no preferred primes.

2. What kind of polynomials could define preferred algebraic extensions of rationals? Irreducible polynomials are certainly an attractive candidate since any polynomial reduces to a product of them. One can say that they define the elementary particles of number theory. Irreducible polynomials have integer coefficients having the property that they do not decompose to products of polynomials with rational coefficients. It would be wrong to say that only these algebraic extensions can appear but there is a temptation to say that one can reduce the study of extensions to their study. One can even consider the possibility that string world sheets associated with products of irreducible polynomials are unstable against decay to those characterize irreducible polynomials.

3. What can one say about irreducible polynomials? Eisenstein criterion (`http://tinyurl.com/47kxjz` states following. If $Q(x) = \sum_{k=0,..,n} a_k x^k$ is n:th order polynomial with integer coefficients and with the property that there exists at least one prime dividing all coefficients a_i except a_n and that p^2 does not divide a_0, then Q is irreducible. Thus one can assign one or more preferred primes to the algebraic extension defined by an irreducible polynomial Q of this kind - in fact any polynomial allowing ramification. There are also other kinds of irreducible polynomials since Eisenstein's condition is only sufficient but not necessary.

   Furthermore, in the algebraic extension defined by Q, the prime ideals P having the above mentioned characteristic property decompose to an n:th power of single prime ideal P_i: $P = P_i^n$. The primes are maximally/completely ramified.

   A good illustration is provided by equations $x^2 + 1 = 0$ allowing roots $x_\pm = \pm i$ and equation $x^2 + 2px + p = 0$ allowing roots $x_\pm = -p \pm \sqrt{p}p - 1$. In the first case the ideals associated with $\pm i$ are different. In the second case these ideals are one and the same since $x_+ == -x_- + p$: hence one indeed has ramification. Note that the first example represents also an example of irreducible polynomial, which does not satisfy Eisenstein criterion. In more general case the n conditions on defined by symmetric functions of roots imply that the ideals are one and same when Eisenstein conditions are satisfied.

4. What is so nice that one could readily construct polynomials giving rise to given preferred primes. The complex roots of these polymials could correspond to the points of partonic 2-surfaces carrying fermions and defining the ends of boundaries of string world sheet. It must be however emphasized that the form of the polynomial depends on the choices of the complex coordinate. For instance, the shift $x \rightarrow x+1$ transforms $(x^n-1)/(x-1)$ to a polynomial satisfying the Eisenstein criterion. One should be able to fix allowed coordinate changes in such a way that the extension remains irreducible for all allowed coordinate changes.

   Already the integral shift of the complex coordinate affects the situation. It would seem that only the action of the allowed coordinate changes must reduce to the action of Galois group permuting the roots of polynomials. A natural assumption is that the complex coordinate corresponds to a complex coordinate transforming linearly under subgroup of isometries of the embedding space.

In the general situation one has $P = \prod P_i^{e(i)}$, $e(i) \geq 1$ so that aso now there are prefered primes so that the appearance of preferred primes is completely general phenomenon.

7.10.3 The origin of p-adic length scale hypothesis?

p-Adic length scale hypothesis emerged from p-adic length scale hypothesis. A possible generalization of this hypothesis is that p-adic primes near powers of prime are physically favored. There indeed exists evidence for the realization of 3-adic time scale hierarchies in living matter [?] (`http://tinyurl.com/jbh9m27`) and in music both 2-adicity and 3-adicity could be present: this is discussed in TGD inspired theory of music harmony and genetic code [K33]. See also [L12, L8].

One explanation would be that for preferred primes the number of p-adic space-time sheets representable also as real space-time sheets is maximal. Imagined worlds would be maximally realizable. Preferred p-adic primes would correspond to ramified primes for extensions with the property that the number of realizable imaginations is especially large for them. Why primes satisfying p-adic length scale hypothesis or its generalization would appear as ramified primes for extensions, which are winners in number theoretical evolution?

Also the weak form of NMP (WNMP) applying also to the purely number theoretic form of NMP [K23] might come in rescue here.

1. Entanglement negentropy for a NE [K23] characterized by n-dimensional projection operator is the $log(N_p(n)$ for some p whose power divides n. The maximum negentropy is obtained if the power of p is the largest power of prime divisor of p, and this can be taken as definition of number theoretical entanglement negentropy (NEN). If the largest divisor is p^k, one has $N = k \times log(p)$. The entanglement negentropy per entangled state is $N/n = klog(p)/n$ and is maximal for $n = p^k$. Hence powers of prime are favoured, which means that p-adic length scale hierarchies with scales coming as powers of p are negentropically favored and should be generated by NMP. Note that $n = p^k$ would define a hierarchy of $h_{eff}/h = p^k$. During the first years of h_{eff} hypothesis I believe that the preferred values obey $h_{eff} = r^k$, r integer not far from $r = 2^{11}$. It seems that this belief was not totally wrong.

2. If one accepts this argument, the remaining challenge is to explain why primes near powers of two (or more generally p) are favoured. $n = 2^k$ gives large entanglement negentropy for the final state. Why primes $p = n_2 = 2^k - r$ would be favored? The reason could be following. $n = 2^k$ corresponds to $p = 2$, which corresponds to the lowest level in p-adic evolution since it is the simplest p-adic topology and farthest from the real topology and therefore gives the poorest cognitive representation of real PE as p-adic PE (Note that $p = 1$ makes formally sense but for it the topology is discrete).

3. WNMP [K23, K41] suggests a more feasible explanation. The density matrix of the state to be reduced is a direct sum over contributions proportional to projection operators. Suppose that the projection operator with largest dimension has dimension n. Strong form of NMP would say that final state is characterized by n-dimensional projection operator. WNMP allows “free will” so that all dimensions $n-k$, $k = 0, 1, ...n-1$ for final state projection operator are possible. 1-dimensional case corresponds to vanishing entanglement negentropy and ordinary state function reduction isolating the measured system from external world.

4. The negentropy of the final state per state depends on the value of k. It is maximal if $n-k$ is power of prime. For $n = 2^k = M_k + 1$, where M_k is Mersenne prime $n-1$ gives the maximum negentropy and also maximal p-adic prime available so that this reduction is favoured by NMP. Mersenne primes would be indeed special. Also the primes $n = 2^k - r$ near 2^k produce large entanglement negentropy and would be favored by NMP.
5. This argument suggests a generalization of p-adic length scale hypothesis so that $p = 2$ can be replaced by any prime.

8 What could be the role of complexity theory in TGD?

I have many times wondered what could be the role of chaos theory or better in TGD. In fact, I would prefer to talk about complexity theory since the chaos in the sense as it is used is only apparent and very different from thermodynamical chaos.

Wikipedia article (see `http://tinyurl.com/qexmowa`) gives a nice summary about the history of chaos theory and I repeat only some main points here. Chaos theory has roots already at the end of 18the century by the works of Poincare (non-periodic orbits in 3-body system) and Hadamard (free particle gliding frictionlessly on surface of constant negative curvature, "Hadamard billiard". In this case all trajectories are unstable diverging exponentially from each other: this is characterized by positive Lyapunov exponent.

Chaos theory got is start from ergodic theory (see `http://tinyurl.com/pfcrz4c`) studying dynamical systems with the original motivation coming from statistical physics. For instance, spin glasses are a representative example of non-ergodic system in which the trajectory of point does not go arbitrary near to every point. The study of non-linear differential equations George David Birkhoff, Andrey Nikolaevich Kolmogorov, Mary Lucy Cartwright and John Edensor Littlewood, and Stephen Smale provides was purely mathematical study of chaotic systems. Smale discovered strange attractor at which periodic orbits form a dense set. Chaos theory was formalized around 1950. At this time it was also discovered that finite-D linear systems do not allow chaos.

The emergence of computers meant breakthrough. Much of chaos theory involves repeated iteration of simple mathematical formulas. Edward Lorentz was a pioneer of chaos theory working with weather prediction and accidentally discovered initial value sensitivity. Benard Mandelbrot discovered fractality and Mitchell Feigenbaum the universality of chaos for iteration of functions of real variable.

Chaotic systems are as far from integrable systems as one could imagine: all orbits are cycles in integrable Hamiltonian dynamics. There are good reasons to suspect that TGD Universe is completely integrable classically. Chaos theory however describes also the emergence of complexity through phase transition like steps - period n-tupling and most importantly by period doubling for iteration of maps.

Chaotic (or actually extremely complex and only apparently chaotic) systems seem to be the diametrical opposite of completely integrable systems about which TGD is a possible example. There is however also something common: in completely integrable classical systems all orbits are cyclic and in chaotic systems they form a dense set in the space of orbits. Furthermore, in chaotic systems the approach to chaos occurs via steps as a control parameter is changed. Same would take place in adelic TGD fusing the descriptions of matter and cognition.

In TGD Universe the hierarchy of extensions of rationals inducing finite-dimensional extension of p-adic number fields defines a hierarchy of adelic physics and provides a natural correlate for evolution. Galois groups and ramified primes appear as characterizers of the extensions. The sequences of Galois groups could characterize an evolution by phase transitions increasing the dimension of the extension associated with the coordinates of "world of classical worlds" (WCW) in turn inducing the extension used at space-time and Hilbert space level. WCW decomposes to sectors characterized by Galois groups G_3 of extensions associated with the 3-surfaces at the ends of space-time surface at boundaries of causal diamond (CD) and G_4 characterizing the space-time surface itself. G_3 (G_4) acts on the discretization and induces a covering structure of the 3-surface (space-time surface). If the state function reduction to the opposite boundary of CD involves localization into a sector with fixed G_3, evolution is indeed mapped to a sequence of G_3s.

Also the cognitive representation defined by the intersection of real and p-adic surfaces with coordinates of points in an extension of rationals evolve. The number of points in this representation becomes increasingly complex during evolution. Fermions at partonic 2-surfaces connected by fermionic strings define a tensor network, which also evolves since the number of fermions can change.

The points of space-time surface invariant under non-trivial subgroup of Galois group define singularities of the covering, and the positions of fermions at partonic surfaces could correspond to these singularities - maybe even the maximal ones, in which case the singular points would be rational. There is a temptation to interpret the p-adic prime characterizing elementary particle as a ramified prime of extension having a decomposition similar to that of singularity so that category theoretic view suggests itself.

One also ends up to ask how the number theoretic evolution could select preferred p-adic primes satisfying the p-adic length scale hypothesis as a survivors in number theoretic evolution, and ends up to a vision bringing strongly in mind the notion of conserved genes as analogy for conservation of ramified primes in extensions of extension. $h_{eff}/h = n$ has natural interpretation as divisor of the order of Galois group of extension. The generalization of $\hbar_{gr} = GMm/v_0 = \hbar_{eff}$ hypothesis to other interactions is discussed in terms of number theoretic evolution as increase of G_3, and one ends up to surprisingly concrete vision for what might happen in the transition from prokaryotes to eukaryotes.

8.1 Basic notions of chaos theory

It is good to begin by summarizing the basic concepts of chaos theory. Again Wikipedia article (see `http://tinyurl.com/qexmowa`) gives a more detailed representation and references. Citing Wikipedia freely: Within the apparent randomness of chaotic complex systems there are patterns, constant feedback loops, repetition, self-similarity, fractals, self-organization and there is sensitivity to initial conditions (butterfly effect) implying the loss of predictability although chaotic systems as such are deterministic.

8.1.1 Basic prerequisites for chaotic dynamics

Wikipedia article lists three basic conditions for chaotic dynamics. Dynamics must **a)** be sensitive to initial conditions, **b)** allow topological mixing, **c)** have dense set of periodic orbits.

1. Sensitivity to initial conditions.

   Mathematical formulation for the sensitivity to initial conditions can be formulated by perturbation theory for differential equations. The rate of separation of images of points initially near to each other increases exponentially as $exp(\lambda t)$ in initial value sensitive situation and the approximation fails soon. Lyapunov exponent λ characterizes the time evolution of the difference. In multi-dimensional case there are several Lyapunov exponents but the largest one is often enough to characterize the situation.

2. Topological mixing (transitivity).

   This notion corresponds to everyday intuition about mixing. For instance, the flow defined by a vector field mixes the marker completely with the fluid. Iteration of simple scaling operation is initial value sensitive but does not cause topological mixing. In 1-D case all points larger than one approach to infinity and smaller than 1 to zero so that the behavior is extremely simple.

3. Dense set of periodic orbits.

   Periodic orbits should form a dense set in the space of orbits: every point of space is approached arbitrarily closely by a periodic orbit. In completely integrable system all orbits would be periodic orbits so that the difference of these systems is very delicate and one can wonder whether the conditions a) and b) follow from this delicate difference. One can also ask whether there might be a deep connection between completely integrable and chaotic systems.

   Sharkovkii's theorem states that any 1-D system with dynamics determined by iteration of a continuous function of real argument exhibits a regular cycle of period 3 exhibits all other cycles. This theorem can be generalized further (see `http://tinyurl.com/l7q3rah`).

Introduce Sharkovskii ordering of integers as union of sets consisting of odd integers multiplied by powers of 2. The generalization of the theorem states that if n is a peri89od and precedes k in Sharkovskii ordering then k is prime period (it is not a multiple of smaller period).

The theorem holds true for reals but not for periodic functions at circle which are encountered for iterations defined by powers of cyclic group elements. The discrete subgroup of hyperbolic subgroups of Lie groups do not have not cycles at all.

8.1.2 Strange attractors and Julia sets

Logistic map $x \to kx(1-x)$ is chaotic everywhere but many systems are chaotic only in a subset of phase space. An interesting situation arises when the chaotic behavior takes place at attractor, since all initial positions in the basic of the attractor lead to the attractor and to a chaotic behavior. Lorentz attractor is a well-known example of strange attractor (see Wikipedia article for illustration). It contains dense sets of both periodic and aperiodic orbits.

Julia set (see `http://tinyurl.com/l8jl5ne`) is the boundary of the basin of attraction in chaotic systems defined by iteration of a rational function of complex argument mapping complex plane to itself. Both Julia sets and strange attractors have a fractal structure.

Strange attractors can appear only in spaces with dimension $D \geq 3$. Poincare-Bendixon theorem states that 2-D differential equations on Euclidian plane have very regular behavior. In non-Euclidian geometry situation changes and the hyperbolic character of the geometry implying initial value sensitivity of geodesic motion is the reason for this. Also infinite-D linear systems can exhibit chaotic behavior.

8.2 How to assign chaos/complexity theory with TGD?

Completely integrable systems can be seen as a diametric opposite of chaotic systems. If classical TGD indeed represents a completely integrable system meaning that space-time surfaces as preferred extremals can be constructed explicitly, one might think that chaos theory need not have much to do with classical TGD. Chaos is however the end product of transitions making the system more complex, and it might well be that the understanding about the emergence of complexity in chaotic systems could help to develop the vision about emergence of complexity in TGD. Note also that periodic orbit are dense in chaotic systems so that diametrical opposites might actually meet.

The most relevant TGD based ingredients used in the sequel are following: WCW [K34]; strong form of holography (SH) [K45], quantum classical correspondence (QCC), zero energy ontology (ZEO) [K24], dark matter as hierarchy of phases with effective Planck constant $h_{eff}/h = n$ [K14, ?, K30], p-Adic physics as physics of cognition [K27, K23, K32] [L17], adelic physics [L17] fusing the physics of matter and cognition by integrating reals and extensions of various p-adic number fields induced by an extension of rationals to a larger structure, and the notions of adelic manifold and associated cognitive representation [L10] , Negentropy Maximization Principle (NMP) [K23] satisfied automatically in statistical sense in adelic physics [L17].

8.2.1 Complexity in TGD

Complexity is often taken to mean computational complexity for classical computations. Complexity as it is understood in the sequel relates very closely cognition. Too complex looks chaotic since our cognitive abilities do not allow to discern too complex patterns. Hence complexity theory should characterize cognitive representations whatever they are.

Number theoretic vision about TGD serves as the guideline here.

1. In adelic TGD [K21] cognitive representations correspond to the intersections of real space-time surfaces and their p-adic variants obeying same field equations and representing correlates for cognition. In these intersections the coordinates of points belong to an extension of rationals defining adele [L10].

   One ends up with a generalization of the notion of manifold to adelic manifold. Intersection defines a common discrete spine consisting of points with coordinates in the extension of rationals defining the adele. These poins are shared by the real and p-adic variants of the adelic manifold. I have called this manifold also monadic manifold since there is strong resemblance with the ideas of Leibniz. In real sector this manifold differs from ordinary manifold in that the open sets are labelled by a discrete set of points in the intersection.

In TGD framework it is essential that the spine of the space-time surface consists of points of embedding space for which it is convenient to use preferred coordinates.

2. Complexity corresponds roughly to the dimension of extension of rationals defining the adeles. p-Adic differential equations are non-deterministic due to the existence of p-adic pseudo constants depending on finite number of p-adic digits of the p-adic number. This non-determinism is identified as a correlate for imagination. p-Adic variants of space-time surfaces are not uniquely determined this means finite cognitive resolution.

   By SH [K42] the data associated with string world sheets, partonic 2-surfaces, and discretization allow to construct space-time surfaces as preferred extremals of the action principle defining classical TGD and to find the Kähler function for WCW geometry. It is quite well possible that the data allowing to construct p-adic space-time surfaces does not allow continuation to a preferred extremal: all imaginations are not realizable!

   The algebraic dimension of the extension could be relevant for the ability of mathematical cognition to imagine spaces with dimension higher than that for the real 3-space. Besides the extensions of p-adics induced by algebraic extensions of rationals also those induced by some root of e are algebraically finite-dimensional. One can imagine also other extensions involving transcendentals in real sense but it is not clear whether there are finite dimensional extensions among them. The finiteness of cognition suggests that only these extensions can be allowed. All imaginations are not realizable!

3. Extension is characterized partially by Galois group (see `http://tinyurl.com/mrvqhz2`) acting as automorphisms meaning that Galois group permutes the roots of the n:th order polynomials defining extensions of rationals via their non-rational roots. So called ramified primes (see `http://tinyurl.com/m32nvcz` and `http://tinyurl.com/oh7tgsw`) provide additional characteristics.

   Iteration cycles appearing in complexity theory for iteration of functions and repeated action of an element Galois group defining a finite Abelian group are mathematically similar notions. Now only cycles are present whereas chaotic systems have aperiodic orbits. The cyclic subgroups of Galois group do not seem to have an natural realization as iterative dynamics except in quantum sense meaning that cyclic orbits are replaced with wave functions labelled by number theoretic integer valued "momenta" for the action of the analog of Cartan subgroup as maximal commutative subgroup for the Galois group. The maximal Abelian Galois group is analog of Cartan subgroup for Galois group of algebraic numbers and states are in its irreducible representations.

   Remark: What is interesting that for polynomials with order larger than 4, one cannot write closed analytic expressions for the roots of the polynomials. This obviously means a fundamental limitation on symbolic cognitive representations provided by explicit formulas. The realization of was a huge step in the evolution of mathematics. Could also the emergence of Galois groups with order larger at space-time level than 5 have meant cognitive revolution - probably at much lower level in the hierarchy? Could this relate also to the fact that space-time dimension is $D = 4$ and thus imaginable using 4-D algebraic extension of rationals?

   A possible measure for the cognitive complexity is the dimension of the Galois group of the extension. One can speak also about the complexity of the Galois group itself - the non-Abelianity of Galois group brings in additional complexity. The number of generators and number of relations between them serve as a measure for complexity of Galois group.

   Extension of rationals is also characterized by so called ramified primes and should have a profound physical meaning. p-Adic length scale hypothesis states that physically preferred primes are near powers of 2 and perhaps also other small primes. Could they correspond to ramified primes. Why just these ramified primes would be survivors in the number theoretic evolution, is the fascinating question to be addressed later.

4. The increase of the dimension of extension or complexity of its Galois group corresponds naturally to evolution interpreted as emergence of algebraic complexity and evolutionary paths could be seen as sequences of inclusions for Galois groups. Chaos would correspond to the limit when the extension of rationals approaches to infinite sub-field of algebraic numbers - say maximal Abelian extension of rationals - so that the number of points in the cognitive representation becomes infinite.

   The Galois group of algebraic numbers - the magic Absolute Group - would characterize this limit as a kind of never achievable mathematical enlightenment. A more practical definition would be that external system is experienced as complex, when its number theoretical complexity exceeds that of the conscious observer so that it is impossible to form a faithful cognitive representation about the system. Note that these cognitive representations could be formulated as homomorphisms between Galois groups. This would suggest a rather nice category theoretical picture about cognitive representations in the self hierarchy.

5. Galois group acts on the cognitive representation associated with the space-time sheet and in general gives n-fold covering of the space-time sheet: n is naturally the dimension of the extension and thus a divisor of the order of Galois group since Galois group acts on the discretization and implies n-sheeted structure for it and therefore also for the space-time surface.

   The value of the effective Planck constant assigned with dark matter as phases of ordinary matter $h_{eff}/h = n$ was identified from very beginning as number of sheets for some kind of covering space of embedding space. n would correspond to a divisor for the order of Galois group for discretized embedding space consisting of points with coordinates in extension of rational. The increase of h_{eff} corresponds to the emergence of also cognitive complexity. Physically it is accompanied by the emergence of quantum coherence and non-locality in increasingly long scales.

8.2.2 General vision about evolution as emergence of complexity

Evolution would mean emergence of number theoretical complexity. Evolutionary paths would naturally correspond to sequences of inclusions (note that recent view allows also temporary "de-evolutions" but in statistical sense evolution occurs). There are infinitely many evolutionary pathways of this kind.

There is a strong resemblance with the inclusion sequences of hyper-finite factors of type II_1 (HHFs) for von Neumann algebras [K44] also playing a central role in TGD and assignable to a fractal hierarchy of isomorphic sub-algebras of super-symplectic algebra associated with the isometries of WCW and related Kac-Moody algebras. It is difficult to believe that this could be an accident.

Evolution must mean a discrete time evolution of some kind - most naturally by non-deterministic quantum version of discrete dynamics, which can be deterministic only in statistical sense. By QCC this evolution should have classical correlates at space-time level. ZEO and TGD inspired theory of consciousness, which can be regarded as a generalization of quantum measurement theory in ZEO, is essential in attempts to concretize this intuition.

1. Galois group codes for the complexity and evolution means the emergence of increasingly complex Galois groups assignable to spacetime surface in a sector of WCW for which WCW coordinates are in corresponding extension of rationals. One can say that evolution defines a path in the space of sectors of WCW characterized by Galois groups. Although the space-time dynamics is expected to be integrable, the notion of complexity still has meaning, and ultimate chaos would emerge at the limit of algebraic numbers as extension of rationals.

2. One can assign Galois group G_5 to space-time surface. Suppose that one an assign Galois groups $G_3 \subset G_4$ with the 3-surfaces at the ends of space-time surfaces at boundaries of CD. This point will be discused below in more detail.

3. At quantum level conscious entities - selves - correspond to sequences off steps consisting of unitary evolution followed by a localization in the moduli space of CD. State function reduction to the opposite boundary of CD means death of self and re-incarnation of self

with opposite arrow of time: also this means localization to a definite sector of WCW [L17, L15]. The sequence of pairs of selves and their time reversals associated with the opposite boundaries of CD (, which itself increases in size) defines a candidate for the non-deterministic quantum analog of iteration in complexity theory.

4. There is a temptation to assume that for the passive boundary of CD all 3-surfaces in quantum superposition have same G_3 - the G_3 that emerged in the first state function reduction to the passive boundary when this self was born. G_3 so would be automatically measured observable and sequence of reductions would define a sequence of G_3s analogous to iteration sequence and also to evolution.

   But can one assume that G_3 is measured automatically in the re-incarnation of self as its time-reversal [K2, K21]? Could only some charateristics of G_3 - say order $n = h_{eff}/h$ - be measured? Also ramified primes characteristize extensions and their measurement is also possible and proposed to characterize elementary particles: they do not fix G_3. These uncertainties are not relevant for the general vision.

5. For the active boundary one would have a superposition of 3-surfaces with different Galois groups and the sequence of the steps consisting of unitary evolution followed by a localization in the moduli space of CDs including also a localization in clock time determined by distance between the tips of CD. Also this would give to quantal discrete dynamics. Also now one can wonder whether Galois group is measured or not. If not, one would have a dispersion like process in the space of Galois groups labelling sectors of WCW.

6. Also the evolution of the tensor net defined by fermionic strings connecting the positions of fermions at partonic 2-surfaces would define a discrete dynamics in the space of these networks both at classical and quantum level [L9]. The dynamics of many-fermion states would determine this evolution.

In the sequel this picture is discussed in more detail.

8.2.3 How can one assign an extension of rationals to WCW, embedding space, and a region of space-time surface?

What fixes the extension used at both WCW level, embedding space level, and space-time level? The natural assumption is that the extension used for WCW coordinates induces the extension used at embedding space level and space-time level. At the level of space-time surfaces WCW coordinates appear as moduli (parameters) characterizing preferred extremals and would have values in an extension of rationals characterizing the adele by inducing the extensions of p-adic sectors.

1. The simplest option is that the extension is dictated by WCW. Preferred WCW coordinates - made possible by maximal isometries and fixed apart from the isometries of WCW - are in the extension: this makes the space of allowed 3-surfaces discrete. This in turn induces a constraint on space-time surfaces: WCW coordinates define parameters characterizing the space-time surface as a preferred extremal. One could use also other coordinates of WCW but these would not be optimal as cognitive representations.

   This applies also at the level of embedding space. Contrary to what I first thought, it is not actually absolutely necessary to use preferred space-time coordinates (subset of embedding space coordinates) since cognitive representation depends on coordinates in finite measurement resolution: consider only spherical and Cartesian coordinates with given resolution defining different discretizations. The preferred coordinates would be preferred because they are cognitively optimal.

2. Real embedding space is replaced with a discrete set of points of H with preferred coordinates in an extension of rationals. The direct identification of the points of extension as real numbers with p-adic numbers is extremely discontinuous although it would respect algebraic symmetries. The situation is saved by the lower dimensionality of space-time surfaces for which the set of points with coordinates in extension is discrete and even finite in the generic

case. The surface $x^n + y^n = z^n$ has only one rational point for $n > 2$! $D = 4 < 8$ for space-time surfaces automatically brings in finite measurement resolution and cognitive resolution induced directly from the restriction on WCW parameters.

SH has as data the intersection plus string world sheets (SH). String world sheets are in the intersection of reality and p-adicities defined by rational functions with coefficients of polynomials in extension, and makes sense both in real and p-adic sense. To these initial data one can assign as a preferred extremal of Kähler action a smooth p-adic space-time surface such that each point is contained in an open set consisting of points with p-adic coordinates having norm smaller than some power of p. This extremal is not unique in the p-adic sectors. In real sector it might not exist at all as already discussed.

3. 3-surface is seen as pair of 3-surfaces assigned to the ends of the space-time surface at boundaries of CD. WCW coordinates parameterize this pair and correspond to extension in 4-D sense. These parameters are expected to decompose to sets of parameters characterizing the 3-D members of pair and parameters characterizing the connecting space-time surface unless it is unique. If so, one can assign to the initial and final 3-surfaces subsets of WCW coordinates.

   The extensions associated with the ends of CD would be extensions in 3-D sense and sub-extensions of the extension in 4-D sense. Hence one can say that classical space-time evolution connecting initial and final 3-surfaces can modify the extension, its Galois group, and therefore also $h_{eff}/h = n$. This would be the classical view a about number theoretic evolution and also about quantum critical fluctuation changing the value of $h_{eff}/h = n$.

4. The extension of rationals for WCW coordinates induces the cognitive representation posing constraints of p-adic space-time surfaces. Adelic sub-WCW consisting of preferred extremals inside given CD decomposes to sectors characterized by an extension of rationals and evolution should correspond number theoretically to a path in the space of WCW sectors.

   This is a restriction on p-adic space-time sheets and thus cognition: the larger the number of points in the intersection, the more precise the cognitive representation is. The increase of the dimension of extension implies that the number of points of cognitive representation increases and it becomes more precise. The cognitive abilities of the system evolve. p-Adic pseudo constants allow imagination but also make the representation imprecise in scales below that defined by the cognitive representation. The continuation to smooth p-adic surface would however explain the highly non-trivial fact that we automatically tend to associate continuous structures with discrete data.

5. The fermions at partonic 2-surfaces are at positions for which preferred space-time coordinates are in extension and can be said to actualize the cognitive representation. It turns out that these positions could naturally correspond to the singularities of the space-time surfaces as n-fold covering in the sense that the dimension of the orbit of Galois group would be reduced at these points.

8.2.4 Can one assign the analog of discrete dynamics to TGD at fundamental level?

Could one assign a discrete symbolic dynamics to classical and quantum TGD?

At classical level the dynamics would correspond to space-time surface connecting the boundaries of CD and 3-surfaces at them. As already explained, the WCW coordinates characterizing space-time surface as a preferred extremal correspond to what might be called Galois group in 4-D sense. These coordinates decompose to coordinates characterizing the coordinates at the 3-surfaces at the ends of of space-time at boundaries of CD in extensions characterized by Galois groups in 3-D sense - the initial and final Galois group. The classical evolutionary step would be a step leading from the initial fo final Galois group serving as classical correlate for quantum evolution.

What about quantum level?

1. One expects that zero energy state in general is a superposition of space-time surfaces with different Galois groups in 4-D sense, G_4. The Galois groups in 3-D sense - G_3 - assignable to the ends of space-time surface would be sub-groups of G_4. If the first state function reduction

to the opposite boundary of CD involves a localization to a sector of WCW having same G_3 at passive boundary for all 3-surfaces in the superposition.

Subsequent reductions at opposite boundaries would define evolutionary pathway in the space of Galois groups G_3 leading in statistical sense to the increase of complexity.

2. The original vision was that Negentropy Maximization Principle (NMP) [K23] is needed as a separate principle to guarantee evolution but adelic physics implies it in statistical sense automatically [L17]. There is infinite number of extensions more complex than given one and only finite number of them less complex.

3. At quantum level the basic notion is self. It corresponds to a discrete sequence steps consisting of unitary evolution followed by a localization in the moduli space of CDs. This would correspond to a dispersion in WCW to sectors characterized by different Galois groups G_4 and G_3 associated with the 3-surface at active boundary. As explained, the state function reduction to the opposite boundary of CD analogous to a halting of quantum computation would correspond to a localization to a sector with definite Galois group G_3.

4. These time discrete time evolutions are non-deterministic unlike the dynamical evolutions studied in chaos theory defined by differential equations or iteration of function. The sequence of unitary time evolutions involving localization in the moduli of CD would however give rise to a quantum analog of iteration and one can ask whether the quantum counterparts for the notions of cycle, super-stable cycle etc... could make sense for the quantum superpositions of 4-surfaces involved. One expects dispersion in the space of Galois groups so that this idea does not look promising. One can also wonder if the sequence of unitary transformations could lead to some kind of asymptotic self-organization pattern before the first state function reduction to the opposite boundary of CD.

It is natural to consider also the evolution of the cognitive representation itself both at the space-time level and forced by the change of the many-fermion state and at quantum level.

1. For a given preferred extremal cognitive representation defines a discrete set of points in an extension of rationals and the number of points in the extension increases as it grows. The positions of fermions at partonic 2-surfaces define the nodes of a graph with strings connecting fermions at different partonic 2-surfaces serving as edges. Evolution of fermionic state changes the topology of this network by adding vertices and changing the connection.

   One can assign a complexity theory to these graphs. A connection with tensor nets [L9] emerging in the description of quantum complexity is highly suggestive. The nodes of the tensor net would correspond to fermions at partonic 2-surfaces. As the number of fermions increases, the complexity of this network increases and also the space-time surface itself becomes more complex. The maximum number of fermions increases with the dimension of extension.

   An interesting proposal is that fermion lines are accompanied by magnetic flux tubes taking the role of wormholes in ER-EPR correspondence (see `http://tinyurl.com/hzqlo6r`), which emerged more than half decade after its TGD analog. The discrete evolution of many-fermion state in state function reductions in the fermionic sector induces the evolution of this network.

2. In the case of graphs one can speak about various kinds of cycles, in particular Hamiltonian cycles going through all points of graph and having no self-intersections. Interestingly, Hamiltonian cycles for icosahedron (here the isometry group of icosahedron is involved as an additional structure) lead to a vision about genetic code and music harmonies [L3].

3. An interesting question concerns the extensions of rationals having as Galois group the isometry groups of Platonic solids: they probably exist. One can also consider the counterparts of Galois groups as discrete subgroups of the Galois group $SO(3)$ of quaternions. They emerge naturally for algebraic discretizations of M^4 regarded as a subspace of complexified quaternions with time axis identified as the real axis for quaternions (for $M^8 - H$ correspondence [K37, K43] see `http://tinyurl.com/mdvazmr`). Platonic solids correspond to

finite discretizations with finite isometry groups belonging to a hierarchy of finite discrete subgroups of $SO(3)$ labelling the hierarchy of inclusions of HFFs: a connection between HFFs and quaternions is suggestive. For HFFs Platonic solids are in unique role in the sense that only for them the action of $SO(3)$ is genuinely 3-D. In Mac Kay correspondence they correspond to exceptional groups.

For this generalization evolution would correspond to evolution in the space of Galois groups for finite-dimensional extensions of rational valued quaternions. p-Adic quaternions do not however form a field since p-adic quaternion can have vanishing norm squared.

4. The wave functions in the Galois group G reduce to wave functions in its coset space G/H if they are invariant under subgroup H. One can also perform the analog of second quantization for fermions in Galois group labelling the space-time sheets (or those of 3-space). In the model of harmony based on Hamilton's cycles the notes of 12-note scale would correspond to vertices of icosahedron obtained as coset space of I/Z_5, where I is icosahedral group with 60 elements. 3-chords of the harmony for a given Hamiltonian cycle would correspond to faces, which are triangles. Single particle fermion states localized at vertices (points of coset space) would correspond to notes of the scale and 3-fermion states localized at vertices of triangle to allowed 3-chords. The observation that one can understand the degeneracies of vertebrate genetic code by introducing besides icosahedron also tetrahedron suggests that both music and genetic code could relate directly to cognition described number theoretically.

5. It is also known that graphs can be identified as representations for Boolean statements (see `http://tinyurl.com/myrkhny`). Many-fermion states represent in TGD framework quantum Boolean statements with fermion number taking the role of bit. Could it be that this graphs indeed represent entanglement many-fermion states having interpretation as quantum Boolean statements?

Can one imagine a quantum counterpart of iteration cycle? The space-time sheets can be seen as covering spaces with the number of sheets equal to the order $n = h_{eff}/h$ of Galois group. This gives additional discrete degrees of freedom and one could have wave functions in Galois group and also in its cyclic subgroup. These might serve as quantum counterparts for iteration cycles. An open question is whether n is always accompanied by $1/n$ fractionization of quantum numbers so that dark elementary particles would have same quantum numbers as ordinary ones but could be said to decompose to n pieces corresponding to sheets of covering.

One can also imagine that the cycles appear in the statistical description. At this limit one obtains deterministic kinetic equations and by their non-linearity one expects that they exhibit chaotic behavior in the usual sense.

8.2.5 Why would primes near powers of two (or small primes) be important?

p-Adic length scale hypothesis states that physically preferred p-adic primes come as primes near prime powers of two and possibly also other small primes. Does this have some analog to complexity theory, period doubling, and with the super-stability associated with period doublings?

Also ramified primes characterize the extension of rationals and would define naturally preferred primes for a given extension.

1. Any rational prime p can be decomposes to a product of powers P_i^k of primes P_i of extension given by $p = \prod_i P_i^{k_i}$, $\sum k_i = n$. If one has $k_i \neq 1$ for some i, one has ramified prime. Prime p is Galois invariant but ramified prime decomposes to lower-dimensional orbits of Galois group formed by a subset of $P_i^{k_i}$ with the same index k_i . One might say that ramified primes are more structured and informative than un-ramified ones. This could mean also representative capacity.

2. Ramification has as its analog criticality leading to the degenerate roots of a polynomial or the lowering of the rank of the matrix defined by the second derivatives of potential function depending on parameters. The graph of potential function in the space defined by its arguments and parameters if n-sheeted singular covering of this space since the potential has several extrema for given parameters. At boundaries of the n-sheeted structure some

sheets degenerate and the dimension is reduced locally . Cusp catastrophe with 3-sheets in catastrophe region is standard example about this.

Ramification also brings in mind super-stability of n-cycle for the iteration of functions meaning that the derivative of n:th iterate $f(f(...)(x) \equiv f^{n)}(x)$ vanishes. Superstability occurs for the iteration of function $f = ax + bx^2$ for $a = 0$.

3. I have considered the possibility that the n-sheeted coverings of the space-time surface are singular in that the sheet co-incide at the ends of space-time surface or at some partonic 2-surfaces. One can also consider the possibility that only some sheets or partonic 2-surfaces co-incide.

   The extreme option is that the singularities occur only at the points representing fermions at partonic 2-surfaces. Fermions could in this case correspond to different ramified primes. The graph of $w = z^{1/2}$ defining 2-fold covering of complex plane with singularity at origin gives an idea about what would be involved. This option looks the most attractive one and conforms with the idea that singularities of the coverings in general correspond to isolated points. It also conforms with the hypothesis that fermions are labelled by p-adic primes and the connection between ramifications and Galois singularities could justify this hypothesis.

4. Category theorists love structural similarities and might ask whether there might be a morphism mapping these singularities of the space-time surfaces as Galois coverings to the ramified primes so that sheets would correspond to primes of extension appearing in the decomposition of prime to primes of extension.

   Could the singularities of the covering correspond to the ramification of primes of extension? Could this degeneracy for given extension be coded by a ramified prime? Could quantum criticality of TGD favour ramified primes and singularities at the locations of fermions at partonic 2-surfaces?

   Could the fundamental fermions at the partonic surfaces be quite generally localize at the singularities of the covering space serving as markings for them? This also conforms with the assumption that fermions with standard value of Planck constants corresponds to 2-sheeted coverings.

5. What could the ramification for a point of cognitive representation mean algebraically? The covering orbit of point is obtained as orbit of Galois group. For maximal singularity the Galois orbit reduces to single point so that the point is rational. Maximally ramified fermions would be located at rational points of extension. For non-maximal ramifications the number of sheets would be reduced but there would be several of them and one can ask whether only maximally ramified primes are realized. Could this relate at the deeper level to the fact that only rational numbers can be represented in computers exactly.

6. Can one imagine a physical correlate for the singular points of the space-time sheets at the ends of the space-time surface? Quantum criticality as analogy of criticality associated with super-stable cycles in chaos theory could be in question. Could the fusion of the space-time sheets correspond to a phenomenon analogous to Bose-Einstein condensation? Most naturally the condensate would correspond to a fractionization of fermion number allowing to put n fermions to point with same M^4 projection? The largest condensate would correspond to a maximal ramification $p = P_i^n$.

Why ramified primes would tend to be primes near powers of two or of small prime? The attempt to answer this question forces to ask what it means to be a survivor in number theoretical evolution. One can imagine two kinds of explanations.

1. Some extensions are winners in the number theoretic evolution, and also the ramified primes assignable to them. These extensions would be especially stable against further evolution representing analogs of evolutionary fossils. As proposed earlier, they could also allow exceptionally large cognitive representations that is large number of points of real space-time surface in extension.

2. Certain primes as ramified primes are winners in the sense the further extensions conserve the property of being ramified.

   (a) The first possibility is that further evolution could preserve these ramified primes and only add new ramified primes. The preferred primes would be like genes, which are conserved during biological evolution. What kind of extensions of existing extension preserve the already existing ramified primes. One could naïvely think that extension of an extension replaces P_i in the extension for $P_i = Q_{ik}^{k_i}$ so that the ramified primes would remain ramified primes.

   (b) Surviving ramified primes could be associated with a exceptionally large number of extensions and thus with their Galois groups. In other words, some primes would have strong tendency to ramify. They would be at criticality with respect to ramification. They would be critical in the sense that multiple roots appear.

   Can one find any support for this purely TGD inspired conjecture from literature? I am not a number theorist so that I can only go to web and search and try to understand what I found. Web search led to a thesis (see `http://tinyurl.com/mkhrssy`) studying Galois group with prescribed ramified primes.

   The thesis contained the statement that not every finite group can appear as Galois group with prescribed ramification. The second statement was that as the number and size of ramified primes increases more Galois groups are possible for given predetermined ramified primes. This would conform with the conjecture. The number and size of ramified primes would be a measure for complexity of the system, and both would increase with the size of the system.

   (c) Of course, both mechanisms could be involved.

Why ramified primes near powers of 2 would be winners? Do they correspond to ramified primes associated with especially many extension and are they conserved in evolution by subsequent extensions of Galois group. But why? This brings in mind the fact that $n = 2^k$-cycles becomes super-stable and thus critical at certain critical value of the control parameter. Note also that ramified primes are analogous to prime cycles in iteration. Analogy with the evolution of genome is also strongly suggestive.

8.2.6 $h_{eff}/h = n$ hypothesis and Galois groups

The natural hypothesis is that $h_{eff}/h = n$ equals to dimension of the extension of rationals in the case that it gives the number of sheets of the covering assignable to the space-time surfaces. The stronger hypothesis is that $h_{eff}/h = n$ is associated with flux tubes and is proportional to the quantum numbers associated with the ends.

1. The basic idea is that Mother Nature is theoretician friendly. As perturbation theory breaks down, the interaction strength expressible as a product of appropriate charges divided by Planck constant, is reduced in the phase transition $\hbar \rightarrow \hbar_{eff}$.

2. In the case of gravitation $GMm \rightarrow= GMm(h/h_{eff})$. Equivalence Principle is satisfied if one has $\hbar_{eff} = \hbar_{gr} = GMm/v_0$, where v_0 is parameter with dimensions of velocity and of the order of some rotation velocity associated with the system. If the masses move with relativistic velocities the interaction strength is proportional to the inner product of four-momenta and therefore to Lorentz boost factors for energies in the rest system of the entire system. In this case one must assume quantization of energies to satisfy the constraint or a compensating reduction of v_0. Interactions strength becomes equal to $\beta_0 = v_0/c$ having no dependence on the masses: this brings in mind the universality associated with quantum criticality.

3. The hypothesis applies to all interactions. For electromagnetism one would have the replacements $Z_1Z_2\alpha \rightarrow Z_1Z_2\alpha(h/h_{em})$ and $\hbar_{em} = Z_1Z_2\alpha/\beta_0$ giving Universal interaction strength. In the case of color interactions the phase transition would lead to the emergence of hadron and it could be that inside hadrons the valence quark have $h_{eff}/h = n > 1$. In this case

one could consider a generalization in which the product of masses is replaced with the inner product of four-momenta. In this case quantization of energy at either or both ends is required. For astrophysical energies one would have effective energy continuum.

This hypothesis suggests the interpretation of $h_{eff}/h = n$ as either the dimension of the extension or the order of its Galois group. If the extensions have dimensions n_1 and n_2, then the composite system would be n_2-dimensional extension of n_1-dimensional extension and have dimension $n_1 \times n_2$. This could be also true for the orders of Galois groups. This would be the case if Galois group of the entire system is free group generated by the G_1 and G_2. One just takes all products of elements of G_1 and G_2 and assumes that they commute to get $G_1 \times G_2$.

Consider gravitation as example.

1. The dimension of the extension should coincide with $\hbar_{eff}/\hbar = n = \hbar_{gr}/\hbar = GMm/v_0\hbar$. The transition occurs only if the value of $\hbar_{gr}/\hbar$ is larger than one. One can say that the dimension of the extension is proportional the product of masses using as unit Planck mass. Rather large extensions are involved and the number of sheets in the Galois covering is huge.

   Note that it is difficult to say how larger Planck constants are actually involved since by gravitational binding the classical gravitational forces are additive and by Equivalence principle same potential is obtained as sum of potentials for splitting of masses into pieces. Also the gravitational Compton length $\lambda_{gr} = GM/v_0$ for m does not depend on m at all so that all particles have same $\lambda_{gr} = GM/v_0$ irrespective of mass (note that v_0 is expressed using units with $c = 1$).

   The maximally incoherent situation would correspond to ordinary Planck constant and the usual view about gravitational interaction between particles. The extreme quantum coherence would mean that both M and m behave as single quantum unit. In many-sheeted space-time this could be understood in terms of a picture based on flux tubes. The interpretation for the degree of coherence is discussed in terms of flux tube connections mediating gravitational flux is discussed in [**?**].

2. h_{gr}/h would be the dimension of the extension, and there is a temptation to associate with the product of masses the product $n = n_1 n_2$ of dimensions n_i associated masses M and m at least in some situations.

   The problem is that the dimension of the extension associated with m would be smaller than 1 for masses $m < m_P/\sqrt{\beta_0}$. Planck mass is about 1.3×10^{19} proton masses and corresponds to a blob of water with size scale 10^{-4} meters - size scale of a large neuron so that only above these scale gravitational quantum coherence would be possible. For $v_0 < 1$ it would seem that even in the case of large neurons one must have more than one neurons. Maybe pyramidal neurons could satisfy the mass constraint and would represent higher level of conscious as compared to other neurons and cells. The giant neurons discovered by the group led by Christof Koch in the brain of of mouse having axonal connections distributed over the entire brain might fulfil the constraint (see `http://tinyurl.com/gvwggsc`).

3. It is difficult to avoid the idea that macroscopic quantum gravitational coherence for multicellular objects with mass at least that for the largest neurons could be involved with biology. Multicellular systems can have mass above this threshold for some critical cell number. This might explain the dramatic evolutionary step distinguishing between prokaryotes (mono-cellulars consisting of Archaea and bacteria including also cellular organelles and cells with sub-critical size) and eukaryotes (multi-cellulars).

4. I have proposed an explanation of the fountain effect appearing in super-fluidity and apparently defying the law of gravity. In this case m was assumed to be the mass of ^4He atom in case of super-fluidity to explain fountain effect [**?**]. The above arguments however allow to ask whether anything changes if one allows the blobs of superfluid to have masses coming as a multiple of $m_P/\sqrt{\beta_0}$. One could check whether fountain effect is possible for super-fluid volumes with mass below $m_P/\sqrt{\beta_0}$.

What about h_{em}? In the case of super-conductivity the interpretation of h_{em}/h as product of orders of Galois groups would allow to estimate the number $N = Q/2e$ of Cooper pairs of a minimal blob of super-conducting matter from the condition that the order of its Galois group is larger than integer. The number $N = Q/2e$ is such that one has $2N\sqrt{\alpha/\beta_0} = n$. The condition is satisfied if one has $\alpha/\beta_0 = q^2$, with $q = k/2l$ such that N is divisible by l. The number of Cooper pairs would be quantized as multiples of l. What is clear that em interaction would correspond to a lower level of cognitive consciousness and that the step to gravitation dominated cognition would be huge if the dark gravitational interaction with size of astrophysical systems is involved [K30]. Many-sheeted space-time allows this in principle.

These arguments support the view that quantum information theory indeed closely relates not only to gravitation but also other interactions. Speculations revolving around blackhole, entropy, and holography, and emergence of space would be replaced with the number theoretic vision about cognition providing information theoretic interpretation of basic interactions in terms of entangled tensor networks [L9]. Negentropic entanglement would have magnetic flux tubes (and fermionic strings at them) as topological correlates. The increase of the complexity of quantum states could occur by the "fusion" of Galois groups associated with various nodes of this network as macroscopic quantum states are formed. Galois groups and their representations would define the basic information theoretic concepts. The emergence of gravitational quantum coherence identified as the emergence of multi-cellulars would mean a major step in biological evolution.

9 Why The Non-trivial Zeros Of Riemann Zeta Should Reside At Critical Line?

The following argument shows that the troublesome looking "1/2" in the non-trivial zeros of Riemann zeta can be understood as being necessary to allow a hermitian realization of the radial scaling generator rd/dr at light-cone boundary, which in the radial light-like radial direction corresponds to half-line R^+. Its presence allows unitary inner product and reduces the situation to that for ordinary plane waves on real axis. For preferred extremals strong form of holography poses extremely strong conditions expected to reduce the scaling momenta $s = 1/2 + iy$ to the zeros of zeta at critical line. RH could be also seen as a necessary condition for the existence of super-symplectic representations and thus for the existence of the "World of Classical Worlds" as a mathematically well-defined object. We can thank the correctness of Riemann's hypothesis for our existence!

9.1 What Is The Origin Of The Troublesome 1/2 In Non-trivial Zeros Of Zeta?

Riemann Hypothess (RH) states that the non-trivial (critical) zeros of zeta lie at critical line $s = 1/2$. It would be interesting to know how many physical justifications for why this should be the case has been proposed during years. Probably this number is finite, but very large it certainly is. In Zero Energy Ontology (ZEO) forming one of the cornerstones of the ontology of quantum TGD, the following justification emerges naturally.

1. The "World of Classical Worlds" (WCW) consisting of space-time surfaces having ends at the boundaries of causal diamond (CD), the intersection of future and past directed light-cones times CP_2 (recall that CDs form a fractal hierarchy). WCW thus decomposes to sub-WCWs and conscious experience for the self associated with CD is only about space-time surfaces in the interior of CD: this is a trong restriction to epistemology, would philosopher say.

   Also the light-like orbits of the partonic 2-surfaces define boundary like entities but as surfaces at which the signature of the induced metric changes from Euclidian to Minkowskian. By holography either kinds of 3-surfaces can be taken as basic objects, and if one accepts strong form of holography, partonic 2-surfaces defined by their intersections plus string world sheets become the basic entities.

2. One must construct tangent space basis for WCW if one wants to define WCW Kähler metric and gamma matrices. Tangent space consists of allowed deformations of 3-surfaces at the ends of space-time surface at boundaries of CD, and also at light-like parton orbits extended by field equations to deformations of the entire space-time surface. By strong form

of holography only very few deformations are allowed since they must respect the vanishing of the elements of a sub-algebra of the classical symplectic charges isomorphic with the entire algebra. One has almost 2-dimensionality: most deformations lead outside WCW and have zero norm in WCW metric.

3. One can express the deformations of the space-like 3-surface at the ends of space-time using a suitable function basis. For CP_2 degrees of freedom color partial waves with well defined color quantum numbers are natural. For light-cone boundary $S^2 \times R^+$, where R^+ corresponds to the light-like radial coordinate, spherical harmonics with well defined spin are natural choice for S^2 and for R^+ analogs of plane waves are natural. By scaling invariance in the light-like radial direction they look like plane waves $\psi_s(r) = r^s = exp(us)$, $u = log(r/r_0)$, $s = x + iy$. Clearly, u is the natural coordinate since it replaces R^+ with R natural for ordinary plane waves.

4. One can understand why $Re[s] = 1/2$ is the only possible option by using a simple argument. One has super-symplectic symmetry and conformal invariance extended from 2-D Riemann surface to metrically 2-dimensional light-cone boundary. The natural scaling invariant integration measure defining inner product for plane waves in R^+ is $du = dr/r = dlog(r/r_0)$ with u varying from $-\infty$ to $+\infty$ so that R^+ is effectively replaced with R. The inner product must be same as for the ordinary plane waves and indeed is for $\psi_s(r)$ with $s = 1/2 + iy$ since the inner product reads as

$$\langle s_1, s_2 \rangle \equiv \int_0^\infty \overline{\psi_{s_1}} \psi_{s_2} dr = \int_0^\infty exp(i(y_1 - y_2) r^{-x_1 - x_2} dr \quad .$$

For $x_1 + x_2 = 1$ one obtains standard delta function normalization for ordinary plane waves:

$$\langle s_1, s_2 \rangle \int \int_{-\infty}^\infty exp[i(y_1 - y_2)u] du \propto \delta(y_1 - y_2) \quad .$$

If one requires that this holds true for all pairs (s_1, s_2), one obtains $x_i = 1/2$ for all s_i. Preferred extremal condition gives extremely powerful additional constraints and leads to a quantisation of $s = -x - iy$: the first guess is that non-trivial zeros of zeta are obtained: $s = 1/2 + iy$. This identification would be natural by generalised conformal invariance. Thus RH is physically extremely well motivated but this of course does not prove it.

5. The presence of the real part $Re[s] = 1/2$ in the eigenvalues of scaling operator apparently breaks hermiticity of the scaling operator. There is however a compensating breaking of hermiticity coming from the fact that real axis is replaced with half-line and origin is pathological. What happens that real part 1/2 effectively replaces half-line with real axis and obtains standard plane wave basis. Note also that the integration measure becomes scaling invariant - something very essential for the representations of super-symplectic algebra. For $Re[s] = 1/2$ the hermicity conditions for the scaling generator rd/dr in R^+ reduce to those for the translation generator d/du in R.

9.2 Relation To Number Theoretical Universality And Existence Of WCW

This relates also to the number theoretical universality and mathematical existence of WCW in an interesting manner.

1. If one assumes that p-adic primes p correspond to zeros $s = 1/2 + y$ of zeta in 1-1 manner in the sense that $p^{iy(p)}$ is root of unity existing in all number fields (algebraic extension of p-adics) one obtains that the plane wave exists for p at points $r = p^n$. p-Adically wave function is discretized to a delta function distribution concentrated at $(r/r_0) = p^n$- a logarithmic lattice. This can be seen as space-time correlate for p-adicity for light-like momenta to be distinguished from that for massive states where length scales come as powers of $p^{1/2}$. Something very similar is obtained from the Fourier transform of the distribution of zeros

at critical line (Dyson's argument), which led to a the TGD inspired vision about number theoretical universality [L5] (see http://tinyurl.com/y7gl4huo).

2. My article "Strategy for Proving Riemann Hypothesis" (http://tinyurl.com/yd7k46ar) [L1] written for 12 years ago ((for a slightly improved version see http://tinyurl.com/ydcfkxwr) relies on coherent states instead of eigenstates of Hamiltonian. The above approach in turn absorbs the problematic 1/2 to the integration measure at light cone boundary and conformal invariance is also now central.

3. Quite generally, I believe that conformal invariance in the extended form applying at metrically 2-D light-cone boundary (and at light-like orbits of partonic 2-surfaces) could be central for understanding why physics requires RH and maybe even for proving RH assuming it is provable at all in existing standard axiomatic system. For instance, the number of generating elements of the extended supersymplectic algebra is infinite (rather than finite as for ordinary conformal algebras) and generators are labelled by conformal weights defined by zeros of zeta (perhaps also the trivial conformal weights). $s = 1/2 + iy$ guarantees that the real parts of conformal weights for all states are integers. By conformal confinement the sum of ys vanishes for physical states. If some weight is not at critical line the situation changes. One obtains as net conformal weights all multiples of x shifted by all half odd integer values. And of course, the realisation as plane waves at boundary of light-cone fails and the resulting loss of unitary makes things too pathological and the mathematical existence of WCW is threatened.

4. The existence of non-trivial zeros outside the critical line could thus spoil the representations of super-symplectic algebra and destroy WCW geometry. RH would be crucial for the mathematical existence of the physical world! And the physical worlds exist only as mathematical objects in TGD based ontology: there are no physical realities behind the mathematical objects (WCW spinor fields) representing the quantum states. TGD inspired theory of consciousness tells that quantum jumps between the zero energy states give rise to conscious experience, and this is in principle all that is needed to understand what we experience.

10 Why Mersenne primes are so special?

Mersenne primes are central in TGD based world view. p-Adic thermodynamics combined with p-adic length scale hypothesis stating that primes near powers of two are physically preferred provides a nice understanding of elementary particle mass spectrum. Mersenne primes $M_k = 2^k - 1$, where also k must be prime, seem to be preferred. Mersenne prime labels hadronic mass scale (there is now evidence from LHC for two new hadronic physics labelled by Mersenne and Gaussian Mersenne), and weak mass scale. Also electron and tau lepton are labelled by Mersenne prime. Also Gaussian Mersennes $M_{G,k} = (1 + i)^k - 1$ seem to be important. Muon is labelled by Gaussian Mersenne and the range of length scales between cell membrane thickness and size of cell nucleus contains 4 Gaussian Mersennes!

What gives Mersenne primes so special physical status? I have considered this problem many times during years. The key idea is that natural selection is realized in much more general sense than usually thought, and has chosen them and corresponding p-adic length scales. Particles characterized by p-adic length scales should be stable in some well-defined sense.

Since evolution in TGD corresponds to generation of information, the obvious guess is that Mersenne primes are information theoretically special. Could the fact that $2^k - 1$ represents almost k bits be of significance? Or could Mersenne primes characterize systems, which are information theoretically especially stable? In the following a more refined TGD inspired quantum information theoretic argument based on stability of entanglement against state function reduction, which would be fundamental process governed by Negentropy Maximization Principle (NMP) and requiring no human observer, will be discussed.

10.1 How to achieve stability against state function reductions?

TGD provides actually several ideas about how to achieve stability against state function reductions. This stability would be of course marvellous fact from the point of view of quantum computation since it would make possible stable quantum information storage. Also living systems could apply this kind of storage mechanism.

1. p-Adic physics leads to the notion of negentropic entanglement (NE) for which number theoretic entanglement entropy is negative and thus measures genuine, possibly conscious information assignable to entanglement (ordinary entanglement entropy measures the lack of information about the state of either entangled system). NMP favors the generation of NE. NE can be however transferred from system to another (stolen using less diplomatically correct expression!), and this kind of transfer is associated with metabolism. This kind of transfer would be the most fundamental crime: biology would be basically criminal activity! Religious thinker might talk about original sin.

   In living matter NE would make possible information storage. In fact, TGD inspired theory of consciousness constructed as a generalization of quantum measurement theory in Zero Energy Ontology (ZEO) identifies the permanent self of living system (replaced with a more negentropic one in biological death, which is also a reincarnation) as the boundary of CD, which is not affected in subsequent state function reductions and carries NE. The changing part of self - sensory input and cognition - can be assigned with opposite changing boundary of CD.

2. Also number theoretic stability can be considered. Suppose that one can assign to the system some extension of algebraic numbers characterizing the WCW coordinates ("world of classical worlds") parametrizing the space-time surface (by strong form of holography (SH) the string world sheets and partonic 2-surfaces continuable to 4-D preferred extremal) associated with it.

   This extension of rationals and corresponding algebraic extensions of p-adic numbers would define the number fields defining the coefficient fields of Hilbert spaces. Assume that you have an entangled system with entanglement coefficients in this number field. Suppose you want to diagonalize the corresponding density matrix. The eigenvalues belong in general case to a larger algebraic extension since they correspond to roots of a characteristic polynomials assignable to the density matrix. Could one say, that this kind of entanglement is stable (at least to some degree) against state function reduction since it means going to an eigenstate which does not belong to the extension used? Reader can decide!

3. Hilbert spaces are like natural numbers with respect to direct sum and tensor product. The dimension of the tensor product is product mn of the dimensions of the tensor factors. Hilbert space with dimension n can be decomposed to a tensor product of prime Hilbert spaces with dimensions which are prime factors of n. In TGD Universe state function reduction is a dynamical process, which implies that the states in state spaces with prime valued dimension are stable against state function reduction since one cannot even speak about tensor product decomposition, entanglement, or reduction of entanglement. These state spaces are quantum indecomposable and would be thus ideal for the storage of quantum information!

   Interestingly, the system consisting of k qubits have Hilbert space dimension $D = 2^k$ and is thus maximally unstable against decomposition to $D = 2$-dimensional tensor factors! In TGD Universe NE might save the situation. Could one imagine a situation in which Hilbert space with dimension $M_k = 2^k - 1$ stores the information stably? When information is processed this state space would be mapped isometrically to 2^k-dimensional state space making possible quantum computations using qubits. The outcome of state function reduction halting the computation would be mapped isometrically back to M_k-D space. Note that isometric maps generalizing unitary transformations are an essential element in the proposal for the tensor net realization of holography and error correcting codes [L9]. Can one imagine any concrete realization for this idea? This question be considered in the sequel.

10.2 How to realize $M_k = 2^k - 1$-dimensional Hilbert space physically?

One can imagine at least three physical realizations of $M_k = 2^k - 1$-dimensional Hilbert space.

1. The set with k elements has 2^k subsets. One of them is empty set and cannot be physically realized. Here the reader might of course argue that if they are realized as empty boxes, one

can realize them. If empty set has no physical realization, the wave functions in the set of non-empty subsets with $2^k - 1$ elements define $2^k - 1$-dimensional Hilbert space. If $2^k - 1$ is Mersenne prime, this state state space is stable against state function reductions since one cannot even speak about entanglement!

To make quantum computation possible one must map this state space to 2^k dimensional state space by isometric embedding. This is possible by just adding a new element to the set and considering only wave functions in the set of subsets containing this new element. Now also the empty set is mapped to a set containing only this new element and thus belongs to the state space. One has 2^k dimensions and quantum computations are possible. When the computation halts, one just removes this new element from the system, and the data are stored stably!

2. Second realization relies on k bits represented as spins such that $2^k - 1$ is Mersenne prime. Suppose that the ground state is spontaneously magnetized state with $k + l$ parallel spins, with the l spins in the direction of spontaneous magnetization and stabilizing it. $l > 1$ is probably needed to stabilize the direction of magnetization: $l \leq k$ suggests itself as the first guess. Here thermodynamics and a model for spin-spin interaction would give a better estimate.

   The state with the k spins in direction opposite to that for l spins would be analogous to empty set. Spontaneous magnetization disappears, when a sufficient number of spins is in direction opposite to that of magnetization. Suppose that k corresponds to a critical number of spins in the sense that spontaneous magnetization occurs for this number of parallel spins. Quantum superpositions of $2^k - 1$ states for k spins would be stable against state function reduction also now.

   The transformation of the data to a processable form would require an addition of $m \geq 1$ spins in the direction of the magnetization to guarantee that the state with all k spins in direction opposite to the spontaneous magnetization does not induce spontaneous magnetization in opposite direction. Note that these additional stabilizing spins are classical and their direction could be kept fixed by a repeated state function reduction (Zeno effect). One would clearly have a critical system.

3. Third realization is suggested by TGD inspired view about Boolean consciousness. Boolean logic is represented by the Fock state basis of many-fermion states. Each fermion mode defines one bit: fermion in given mode is present or not. One obtains 2^k states. These states have different fermion numbers and in ordinary positive energy ontology their realization is not possible.

   In ZEO situation changes. Fermionic zero energy states are superpositions of pairs of states at opposite boundaries of CD such that the total quantum numbers are opposite. This applies to fermion number too. This allows to have time-like entanglement in which one has superposition of states for which fermion numbers at given boundary are different. This kind of states might be realized for super-conductors to which one at least formally assigns coherent state of Cooper pairs having ill-defined fermion number.

   Now the non-realizable state would correspond to fermion vacuum analogous to empty set. Reader can of course argue that the bosonic degrees of freedom assignable to the space-time surface are still present. I defend this idea by saying that the purely bosonic state might be unstable or maybe even non-realizable as vacuum state and remind that also bosons in TGD framework consists of pairs of fundamental fermions.

   If this state is effectively decoupled from the rest of the Universe, one has $2^k - 1$-dimensional state space and states are stable against state function reduction. Information processing becomes possible by adding some positive energy fermions and corresponding negative energy fermions at the opposite boundaries of CD. Note that the added fermions do not have time-like quantum entanglement and do not change spin direction during time evolution.

   The proposal is that Boolean consciousness is realized in this manner and zero energy states represents quantum Boolean thoughts as superposition of pairs $(b_1 \otimes b_2)$ of positive and negative energy states and having identification as Boolean statements $b_1 \rightarrow b_2$. The mechanism would allow both storage of thoughts as memories and their processing by introducing the additional fermion.

10.3 Why Mersenne primes would be so special?

Returning to the original question "Why Mersenne primes are so special?". A possible explanation is that elementary particle or hadron characterized by a p-adic length scale $p = M_k = 2^k - 1$ both stores and processes information with maximal effectiveness. This would not be surprising if p-adic physics defines the physical correlates of cognition assumed to be universal rather than being restricted to human brain.

In adelic physics p-dimensional Hilbert space could be naturally associated with the p-adic adelic sector of the system. Information storage could take place in $p = M_k = 2^k - 1$ phase and information processing (cognition) would take place in 2^k-dimensional state space. This state space would be reached in a phase transition $p = 2^k - 1 \to 2$ changing effective p-adic topology in real sector and genuine p-adic topology in p-adic sector and replacing padic length scale $\propto \sqrt{p} \simeq 2^{k/2}$ with k-nary 2-adic length scale $\propto 2^{k/2}$.

Electron is characterized by the largest not completely super-astrophysical Mersenne prime M_{127} and corresponds to $k = 127$ bits. Intriguingly, the secondary p-adic time scale of electron corresponds to .1 seconds defining the fundamental biorhythm of 10 Hz.

This proposal suffers from deficiencies. It does not explain why Gaussian Mersennes are also special. Gaussian Mersennes correspond ordinary primes near power of 2 but not so near as Mersenne primes do. Neither does it explain why also more general primes $p \simeq 2^k$ seem to be preferred. Furthermore, p-adic length scale hypothesis generalizes and states that primes near powers of at least small primes q: $p \simeq q^k$ are special at least number theoretically. For instance, $q = 3$ seems to be important for music experience and also $q = 5$ might be important (Golden Mean)

Could it be that the proposed model relying on criticality generalizes. There would be $p < 2^k$-dimensional state space allowing isometric embedding to 2^k-dimensional space such that the bit configurations orthogonal to the image would be unstable in some sense. Say against a phase transition changing the direction of magnetization. One can imagine the variants of above described mechanism also now. For $q > 2$ one should consider pinary digits instead of bits but the same arguments would apply (except in the case of Boolean logic).

10.4 Brain and Mersenne integers

I received a link to an interesting the article "Brain Computation Is Organized via Power-of-Two-Based Permutation Logic" by Kun Xie, Grace E. Fox, Jun Liu, Cheng Lyu, Jason C. Lee, Hui Kuang, Stephanie Jacobs, Meng Li, Tianming Liu, Sen Song and Joe Z. Tsien in Frontiers in Systems Neuroscience [?]see `http://tinyurl.com/zfymqrq`).

The proposed model is about how brain classifies neuronal inputs. The following represents my attempt to understand the model of the article.

1. One can consider a situation in which one has n inputs identifiable as bits: bit could correspond to neuron firing or not. The question is however to classify various input combinations. The obvious criterion is how many bits are equal to 1 (corresponding neuron fires). The input combinations in the same class have same number of firing neurons and the number of subsets with k elements is given by the binomial coefficient $B(n,k) = n!/k!(n-k)!$. There are clearly $n-1$ different classes in the classification since no neurons firing is not a possible observation. The conceptualization would tell how many neurons fire but would not specify which of them.

2. To represent these bit combinations one needs $2^n - 1$ neuron groups acting as unit representing one particular firing combination. These subsets with k elements would be mapped to neuron cliques with k firing neutrons. For given input individual firing neurons ($k = 1$) would represent features, lowest level information. The n cliques with $k = 2$ neurons would represent a more general classification of input. One obtains $M_n = 2^n - 1$ combinations of firing neurons since the situations in which no neurons are firing is not counted as an input.

3. If all neurons are firing then all the however level cliques are also activated. Set theoretically the subsets of set partially ordered by the number of elements form an inclusion hierarchy, which in Boolean algebra corresponds to the hierarchy of implications in opposite direction. The clique with all neurons firing correspond to the most general statement implying all the lower level statements. At k:th level of hierarchy the statements are inconsistent so that one has $B(n,k)$ disjoint classes.

The $M_n = 2^n - 1$ (Mersenne number) labelling the algorithm is more than familiar to me.

1. For instance, electron's p-adic prime corresponds to Mersenne prime $M_{127} = 2^{127} - 1$, the largest not completely super-astrophysical Mersenne prime for which the mass of particle would be extremely small. Hadron physics corresponds to M_{107} and M_{89} to weak bosons and possible scaled up variant of hadron physics with mass scale scaled up by a factor 512 ($= 2^{(107-89)/2}$). Also Gaussian Mersennes seem to be physically important: for instance, muon and also nuclear physics corresponds to $M_{G,n} = (1+i)^n - 1$, $n = 113$.

2. In biology the Mersenne prime $M_7 = 2^7 - 1$ is especially interesting. The number of statements in Boolean algebra of 7 bits is 128 and the number of statements that are consistent with given atomic statement (one bit fixed) is $2^6 = 64$. This is the number of genetic codons which suggests that the letters of code represent 2 bits. As a matter of fact, the so called Combinatorial Hierarchy $M(n) = M_{M(n-1)}$ consists of Mersenne primes $n = 3, 7, 127, 2^{127}-1$ and would have an interpretation as a hierarchy of statements about statements about ... It is now known whether the hierarchy continues beyond M_{127} and what it means if it does not continue. One can ask whether M_{127} defines a higher level code - memetic code as I have called it - and realizable in terms of DNA codon sequences of 21 codons [L8] (see `http://tinyurl.com/jukyq6y`).

3. The Gaussian Mersennes $M_{G,n}$ $n = 151, 157, 163, 167$, can be regarded as a number theoretical miracles since the these primes are so near to each other. They correspond to p-adic length scales varying between cell membrane thickness 10 nm and cell nucleus size 2.5 μm and should be of fundamental importance in biology. I have proposed that p-adically scaled down variants of hadron physics and perhaps also weak interaction physics are associated with them.

I have made attempts to understand why Mersenne primes M_n and more generally primes near powers of 2 seem to be so important physically in TGD Universe.

1. The states formed from n fermions form a Boolean algebra with 2^n elements, but one of the elements is vacuum state and could be argued to be non-realizable. Hence Mersenne number $M_n = 2^n - 1$. The realization as algebra of subsets contains empty set, which is also physically non-realizable. Mersenne primes are especially interesting as sine the reduction of statements to prime nearest to M_n corresponds to the number $M_n - 1$ of physically representable Boolean statements.

2. Quantum information theory suggests itself as explanation for the importance of Mersenne primes since M_n would correspond the number of physically representable Boolean statements of a Boolean algebra with n-elements. The prime $p \leq M_n$ could represent the number of elements of Boolean algebra representable p-adically [L12] (see `http://tinyurl.com/gp9mspa`).

3. In TGD Fermion Fock states basis has interpretation as elements of quantum Boolean algebra and fermionic zero energy states in ZEO expressible as superpositions of pairs of states with same net fermion numbers can be interpreted as logical implications. WCW spinor structure would define quantum Boolean logic as "square root of Kähler geometry". This Boolean algebra would be infinite-dimensional and the above classification for the abstractness of concept by the number of elements in subset would correspond to similar classification by fermion number. One could say that bosonic degrees of freedom (the geometry of 3-surfaces) represent sensory world and spinor structure (many-fermion states) represent that logical thought in quantum sense.

4. Fermion number conservation would seem to represent an obstacle but in ZEO it can circumvented since zero energy states can be superpositions of pair of states with opposite fermion number F at opposite boundaries of causal diamond (CD) in such a way that F varies. In state function reduction however localization to single value of F is expected to happen usually. If superconductors carry coherent states of Cooper pairs, fermion number for them is ill defined and this makes sense in ZEO but not in standard ontology unless one gives up the super-selection rule that fermion number of quantum states is well-defined.

One can of course ask whether primes n defining Mersenne primes (see `http://tinyurl.com/l3lxe2n`) could define preferred numbers of inputs for subsystems of neurons. This would predict $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257, ..$ define favoured numbers of inputs. $n = 127$ would correspond to memetic code.

11 Number Theoretical Feats and TGD Inspired Theory of Consciousness

Number theoretical feats of some mathematicians like Ramanujan remain a mystery for those believing that brain is a classical computer. Also the ability of idiot savants - lacking even the idea about what prime is - to factorize integers to primes challenges the idea that an algorithm is involved. In this article I discuss ideas about how various arithmetical feats such as partitioning integer to a sum of integers and to a product of prime factors might take place. The ideas are inspired by the number theoretic vision about TGD suggesting that basic arithmetics might be realized as naturally occurring processes at quantum level and the outcomes might be "sensorily perceived". One can also ask whether zero energy ontology (ZEO) could allow to perform quantum computations in polynomial instead of exponential time.

The indian mathematician Srinivasa Ramanujan is perhaps the most well-known example about a mathematician with miraculous gifts. He told immediately answers to difficult mathematical questions - ordinary mortals had to hard computational work to check that the answer was right. Many of the extremely intricate mathematical formulas of Ramanujan have been proved much later by using advanced number theory. Ramanujan told that he got the answers from his personal Goddess. A possible TGD based explanation of this feat relies on the idea that in zero energy ontology (ZEO) quantum computation like activity could consist of steps consisting quantum computation and its time reversal with long-lasting part of each step performed in reverse time direction at opposite boundary of causal diamond so that the net time used would be short at second boundary.

The adelic picture about state function reduction in ZEO suggests that it might be possible to have direct sensory experience about prime factorization of integers [L11]. What about partitions of integers to sums of primes? For years ago I proposed that symplectic QFT is an essential part of TGD. The basic observation was that one can assign to polygons of partonic 2-surface - say geodesic triangles - Kähler magnetic fluxes defining symplectic invariance identifiable as zero modes. This assignment makes sense also for string world sheets and gives rise to what is usually called Abelian Wilson line. I could not specify at that time how to select these polygons. A very natural manner to fix the vertices of polygon (or polygons) is to assume that they correspond ends of fermion lines which appear as boundaries of string world sheets. The polygons would be fixed rather uniquely by requiring that fermions reside at their vertices.

The number 1 is the only prime for addition so that the analog of prime factorization for sum is not of much use. Polygons with $n = 3, 4, 5$ vertices are special in that one cannot decompose them to non-degenerate polygons. Non-degenerate polygons also represent integers $n > 2$. This inspires the idea about numbers $\{3, 4, 5\}$ as "additive primes" for integers $n > 2$ representable as non-degenerate polygons. These polygons could be associated many-fermion states with negentropic entanglement (NE) - this notion relate to cognition and conscious information and is something totally new from standard physics point of view. This inspires also a conjecture about a deep connection with arithmetic consciousness: polygons would define conscious representations for integers $n > 2$. The splicings of polygons to smaller ones could be dynamical quantum processes behind arithmetic conscious processes involving addition.

11.1 How Ramanujan did it?

Lubos Motl wrote recently a blog posting (http://tinyurl.com/zduu72p) about $P \neq NP$ computer in the theory of computation based on Turing's work. This unproven conjecture relies on a classical model of computation developed by formulating mathematically what the women doing the hard computational work in offices at the time of Turing did. Turing's model is extremely beautiful mathematical abstraction of something very every-daily but does not involve fundamental physics in any manner so that it must be taken with caution. The basic notions include those of algorithm and recursive function, and the mathematics used in the model is mathematics of integers. Nothing is assumed about what conscious computation is and it is somewhat ironic that this model has been taken by strong AI people as a model of consciousness!

1. A canonical model for classical computation is in terms of Turing machine, which has bit sequence as inputs and transforms them to outputs and each step changes its internal state. A more concrete model is in terms of a network of gates representing basic operations for the incoming bits: from this basic functions one constructs all recursive functions. The computer and program actualize the algorithm represented as a computer program and eventually halts - at least one can hope that it does so. Assuming that the elementary operations require some minimum time, one can estimate the number of steps required and get an estimate for the dependence of the computation time as function of the size of computation.

2. If the time required by a computation, whose size is characterized by the number N of relevant bits, can be carried in time proportional to some power of N and is thus polynomial, one says that computation is in class P. Non-polynomial computation in class NP would correspond to a computation time increasing with N faster than any power of N, say exponentially. Donald Knuth, whose name is familiar for everyone using Latex to produce mathematical text, believes on $P = NP$ in the framework of classical computation. Lubos in turn thinks that the Turing model is probably too primitive and that quantum physics based model is needed and this might allow $P = NP$.

What about quantum computation as we understand it in the recent quantum physics: can it achieve $P = NP$?

1. Quantum computation is often compared to a superposition of classical computations and this might encourage to think that this could make it much more effective but this does not seem to be the case. Note however that the amount of information represents by N qubits is however exponentially larger than that represented by N classical bits since entanglement is possible. The prevailing wisdom seems to be that in some situations quantum computation can be faster than the classical one but that if $P = NP$ holds true for classical computation, it holds true also for quantum computations. Presumably because the model of quantum computation begins from the classical model and only (quantum computer scientists must experience this statement as an insult - apologies!) replaces bits with qubits.

2. In quantum computer one replaces bits with entangled qubits and gates with quantum gates and computation corresponds to a unitary time evolution with respect to a discretized time parameter constructed in terms of fundamental simple building bricks. So called tensor networks realize the idea of local unitary in a nice manner and has been proposed to defined error correcting quantum codes. State function reduction halts the computation. The outcome is non-deterministic but one can perform large number of computations and deduce from the distribution of outcomes the results of computation.

What about conscious computations? Or more generally, conscious information processing. Could it proceed faster than computation in these sense of Turing? To answer this question one must first try to understand what conscious information processing might be. TGD inspired theory of consciousnesss provides one a possible answer to the question involving not only quantum physics but also new quantum physics.

1. In TGD framework Zero energy ontology (ZEO) replaces ordinary positive energy ontology and forces to generalize the theory of quantum measurement. This brings in several new

elements. In particular, state function reductions can occur at both boundaries of causal diamond (CD), which is intersection of future and past direct light-cones and defines a geometric correlate for self. Selves for a fractal hierarchy - CDs within CDs and maybe also overlapping. Negentropy Maximization Principle (NMP) is the basic variational principle of consciousness and tells that the state function reductions generate maximum amount of conscious information. The notion of negentropic entanglement (NE) involving p-adic physics as physics of cognition and hierarchy of Planck constants assigned with dark matter are also central elements.

2. NMP allows a sequence of state function reductions to occur at given boundary of diamond-like CD - call it passive boundary. The state function reduction sequence leaving everything unchanged at the passive boundary of CD defines self as a generalized Zeno effect. Each step shifts the opposite - active - boundary of CD "upwards" and increases its distance from the passive boundary. Also the states at it change and one has the counterpart of unitary time evolution. The shifting of the active boundary gives rise to the experienced time flow and sensory input generating cognitive mental images - the "Maya" aspect of conscious experienced. Passive boundary corresponds to permanent unchanging "Self".

3. Eventually NMP forces the first reduction to the opposite boundary to occur. Self dies and reincarnates as a time reversed self. The opposite boundary of CD would be now shifting "downwards" and increasing CD size further. At the next reduction to opposite boundary re-incarnation of self in the geometric future of the original self would occur. This would be re-incarnation in the sense of Eastern philosophies. It would make sense to wonder whose incarnation in geometric past I might represent!

Could this allow to perform fast quantal computations by decomposing the computation to a sequence in which one proceeds in both directions of time? Could the incredible feats of some "human computers" rely on this quantum mechanism (see `http://tinyurl.com/hk5baty`). The indian mathematician Srinivasa Ramanujan (see `http://tinyurl.com/l42q7a2`) is the most well-known example of a mathematician with miraculous gifts. He told immediately answers to difficult mathematical questions - ordinary mortals had to hard computational work to check that the answer was right. Many of the extremely intricate mathematical formulas of Ramanujan have been proved much later by using advanced number theory. Ramanujan told that he got the answers from his personal Goddess.

Might it be possible in ZEO to perform quantally computations requiring classically non-polynomial time much faster - even in polynomial time? If this were the case, one might at least try to understand how Ramanujan did it although higher levels selves might be involved also (did his Goddess do the job?).

1. Quantal computation would correspond to a state function reduction sequence at fixed boundary of CD defining a mathematical mental image as sub-self. In the first reduction to the opposite boundary of CD sub-self representing mathematical mental image would die and quantum computation would halt. A new computation at opposite boundary proceeding to opposite direction of geometric time would begin and define a time-reversed mathematical mental image. This sequence of reincarnations of sub-self as its time reversal could give rise to a sequence of quantum computation like processes taking less time than usually since one half of computations would take place at the opposite boundary to opposite time direction (the size of CD increases as the boundary shifts).

2. If the average computation time is same at both boundaries, the computation time would be only halved. Not very impressive. However, if the mental images at second boundary - call it A - are short-lived and the selves at opposite boundary B are very long-lived and represent very long computations, the process could be very fast from the point of view of A! Could one overcome the $P \neq NP$ constraint by performing computations during time-reversed re-incarnations?! Short living mental images at this boundary and very long-lived mental images at the opposite boundary - could this be the secret of Ramanujan?

3. Was the Goddess of Ramanujan - self at higher level of self-hierarchy - nothing but a time reversal for some mathematical mental image of Ramanujan (Brahman=Atman!), representing

very long quantal computations! We have night-day cycle of personal consciousness and it could correspond to a sequence of re-incarnations at some level of our personal self-hierarchy. Ramanujan tells that he met his Goddess in dreams. Was his Goddess the time reversal of that part of Ramanujan, which was unconscious when Ramanujan slept? Intriguingly, Ramanujan was rather short-lived himself - he died at the age of 32! In fact, many geniuses have been rather short-lived.

4. Why the alter ego of Ramanujan was Goddess? Jung intuited that our psyche has two aspects: anima and animus. Do they quite universally correspond to self and its time reversal? Do our mental images have gender?! Could our self-hierarchy be a hierarchical collection of anima and animi so that gender would be something much deeper than biological sex! And what about Yin-Yang duality of Chinese philosophy and the ka as the shadow of persona in the mythology of ancient Egypt?

11.2 Symplectic QFT, $\{3, 4, 5\}$ as Additive Primes, and Arithmetic Consciousness

For years ago I proposed that symplectic QFT is an essential part of TGD [K8, K37]. The basic observation was that one can assign to polygons of partonic 2-surface - say geodesic triangles - Kähler magnetic fluxes defining symplectic invariance identifiable as zero modes. This assignment makes sense also for string world sheets and gives rise to what is usually called Abelian Wilson line. I could not specify at that time how to select these polygons in the case of partonic 2-surfaces.

The recent proposal of Maldacena and Arkani-Hamed [B2] (see `http://tinyurl.com/ych26gcm`) that CMB might contain signature of inflationary cosmology as triangles and polygons for which the magnitude of n-point correlation function is enhanced led to a progress in this respect. In the proposal of Maldacena and Arkani-Hamed the polygons are defined by momentum conservation. Now the polygons would be fixed rather uniquely by requiring that fermions reside at their vertices and momentum conservation is not involved.

This inspires the idea about numbers $\{3, 4, 5\}$ as "additive primes" for integers $n > 2$ representable as non-degenerate polygons. Geometrically one could speak of prime polygons not decomposable to lower non-degenerate polygons. These polygons are different from those of Maldacena and Arkani-Hamed and would be associated many-fermion states with negentropic entanglement (NE) - this notion relates to cognition and conscious information and is something totally new from standard physics point of view. This inspires also a conjecture about a deep connection with arithmetic consciousness: polygons would define representations for integers $n > 2$. The splicings of polygons to smaller ones could be dynamical quantum processes behind arithmetic conscious processes involving addition. I have already earlier considered a possible counterpart for conscious prime factorization in the adelic framework [L11].

11.2.1 Basic ideas of TGD inspired theory of conscious very briefly

Negentropy Maximization Principle (NMP) is the variational principle of consciousness in TGD framework. It says that negentropy gain in state function reduction (quantum jump re-creating Universe) is maximal. State function reduction is basically quantum measurement in standard QM and sensory qualia (for instance) could be perhaps understood as quantum numbers of state resulting in state function reduction. NMP poses conditions on whether this reduction can occur. In standard ontology it would occur always when the state is entangled: reduction would destroy the entanglement and minimize entanglement entropy. When cognition is brought in, the situation changes.

The first challenge is to define what negentropic entanglement (NE) and negentropy could mean.

1. In real physics without cognition one does not have any definition of negentropy: on must define negentropy as reduction of entropy resulting as conscious entity gains information. This kind of definition is circular in consciousness theory.

2. In p-adic physics one can define number theoretic entanglement entropy with same basic properties as ordinary Shannon entropy. For some p-adic number fields this entropy can be

negative and this motivates an interpretation as conscious information related to entanglement - rather to the ignorance of external observer about entangled state. The prerequisite is that the entanglement probabilities belong to an an extension of rationals inducing a finite-dimensional extension of rationals. Algebraic extensions are such extensions as also those generate by a root of e (e^p is p-adic number in Q_p).

A crucial step is to fuse together sensory and cognitive worlds as different aspects of existence.

1. One must replace real universe with adelic one so that one has real space-time surfaces and their p-adic variants for various primes p satisfying identical field equations. These are related by strong form of holography (SH) in which 2-D surfaces (string world sheets and partonic 2-surfaces) serve as "space-time genes" and obey equations which make sense both p-adically in real sense so that one can identify them as points of "world of classical worlds" (WCW).

2. One can say that these 2-surfaces belong to intersection of realities and p-adicities - intersection of sensory and cognitive. This demands that the parameters appearing in the equations for 2-surface belong algebraic extension of rational numbers: the interpretation is that this hierarchy of extensions corresponds to evolutionary hierarchy. This also explains imagination in terms of the p-adic space-time surfaces which are not so unique as the real one because of inherent non-determinism of p-adic differental equations. What can be imagined cannot be necessarily realized. You can continued p-adic 2-surface to 4-D surface but not to real one.

There is also second key assumption involved.

1. Hilbert space of quantum states is *same* for real and p-adic sectors of adelic world: for instance, tensor product would lead to total nonsense since there would be both real and p-adic fermions. This means same quantum state and same entanglement but seen from sensory and various cognitive perspectives. This is the basic idea of adelicity: the p-adic norms of rational number characterize the norm of rational number. Now various p-adic conscious experiences characterize the quantum state.

2. Real perspective sees entanglement always as entropic. For some finite number number of primes p p-adic entanglement is however negentropic. For instance, for entanglement probabilities $p_i = 1/N$, the primes appearing as factors of N are such information carrying primes. The presence of these primes can make the entanglement stable. The total entropy equal to the sum of negative real negentropy + various p-adic negentropies can be positive and cannot be reduced in the reduction so that reduction does not occur at all! Entanglement is stabilized by cognition and the randomness of state function reduction tamed: matter has power over matter!

3. There is analogy with the reductionism-holism dichotomy. Real number based view is reductionistic: information is obtained when the entangled state is split into un-entangled part. p-Adic number based view is holistic: information is inthe negentropic entanglement and can be seen as abstraction or rule. The superposition of state pairs represents a rule with state pairs (a_i, b_i) representing the instance of the rule A↔B. Maximal entanglement defined by entanglement probabilities $p_i = 1/N$ makes clear the profound distinction between these views. In real sector the negentropy is negative and smallest possible. In p-adic sector the negentropy is maximum for p-adic primes appearing as factors of N and total negentropy as their sum is large. NE allows to select unique state basis if the probabilities p_i are different.

   For $p_i = 1/N$ one can choose any unitary related state basis since unit matrix is invariant under unitary transformations. From the real point of view the ignorance is maximal and entanglement entropy is indeed maximal. For instance, in case of Schrödinger cat one could choose the cat's state basis to be any superposition of dead and alive cat and a state orthogonal to it. From p-adic view information is maximal. The reports of meditators, in particular Zen buddhists, support this interpretation. In "enlightened state" all discriminations disappear: it does not make sense to speak about dead or alive cat or anything between these two options. The state contains information about entire state - not about its parts. It is not information expressible using language relying on making of distinctions but silent wisdom.

11.2.2 How do polygons emerge in TGD framework?

The duality defined by strong form of holography (SH) has 2 sides. Space-time side (bulk) and boundary side (string world sheets and partonic 2-surfaces). 2-D half of SH would suggest a description based on string world sheets and partonic 2-surfaces. This description should be especially simple for the quantum states realized as spinor fields in WCW ("world of classical worlds"). The spinors (as opposed to spinor fields) are now fermionic Fock states assignable to space-time surface defining a point of WCW. TGD extends ordinary 2-D conformal invariance to super-symplectic symmetry applying at the boundary of light-cone: note that given boundary of causal diamond (CD) is contained by light-cone boundary.

1. The correlation functions at embedding space level for fundamental objects, which are fermions at partonic 2-surfaces could be calculated by applying super-symplectic invariance having conformal structure. I have made rather concrete proposals in this respect. For instance, I have suggested that the conformal weights for the generators of supersymplectic algebra are given by poles of fermionic zeta $\zeta_F(s) = \zeta(s)/\zeta(2s)$ and thus include zeros of zeta scaled down by factor 1/2 [K13]. A related proposal is conformal confinement guaranteeing the reality of net conformal weights.

2. The conformally invariant correlation functions are those of super-symplectic CFT at light-cone boundary or its extension to CD. There would be the analog of conformal invariance associated with the light-like radial coordinate r_M and symplectic invariance associated with CP_2 and sphere S^2 localized with respect to r_M analogous to the complex coordinate in ordinary conformal invariance and naturally continued to hypercomplex coordinate at string world sheets carrying the fermionic modes and together with partonic 2-surfaces defining the boundary part of SH.

Symplectic invariants emerge in the following manner. Positive and negative energy parts of zero energy states would also depend on zero modes defined by super-symplectic invariants and this brings in polygons. Polygons emerge also from four-momentum conservation. These of course are also now present and involve the product of Lorentz group and color group assignable to CD near its either boundary. It seems that the extension of Poincare translations to Kac-Moody type symmetry allows to have full Poincare invariance (in its interior CD looks locally like $M^4 \times CP_2$).

1. One can define the symplectic invariants as magnetic fluxes associated with S^2 and CP_2 Kähler forms. For string world sheets one would obtain non-integrable phase factors. The vertices of polygons defined by string world sheets would correspond to the intersections of the string world sheets with partonic 2-surfaces at the boundaries of CD and at partonic 2-surfaces defining generalized vertices at which 3 light-like 3-surfaces meet along their ends.

2. Any polygon at partonic 2-surface would also allow to define such invariants. A physically natural assumption is that the vertices of these polygons are realized physically by adding fermions or antifermions at them. Kähler fluxes can be expressed in terms of non-integrable phase factors associated with the edges. This assumption would give the desired connection with quantum physics and fix highly uniquely but not completely the invariants appearing in physical states.

The correlated polygons would be thus naturally associated with fundamental fermions and a better analogy would be negentropically entangled n-fermion state rather than corresponding to maximum of the modulus of n-point correlation function. Hierarchy of Planck constants makes these states possible even in cosmological scales. The point would be that negentropic entanglement assignable to the p-adic sectors of WCW would be in key role.

11.2.3 Symplectic invariants and Abelian non-integrable phase factors

Consider now the polygons assignable to many-fermion states at partonic 2-surfaces.

1. The polygon associated with a given set of vertices defined by the position of fermions is far from unique and different polygons correspond to different physical situations. Certainly one must require that the geodesic polygon is not self-intersecting and defines a polygon or set of polygons.

2. Geometrically the polygon is not unique unless it is convex. For instance, one can take regular n-gon and add one vertex to its interior. The polygon can be also constructed in several ways. From this one obtains a non-convex $n+1$-gon in $n+1$ ways.

3. Given polygon is analogous with Hamiltonian cycle connecting all points of given graph. Now one does not have graph structure with edges and vertices unless one defines it by nearest neighbor property. Platonic solids provide an example of this kind of situation. Hamiltonian cycles [A1, A4] are key element in the TGD inspired model for music harmony leading also to a model of genetic code [K33] [L3].

4. One should somehow fix the edges of the polygon. For string world sheets the edges would be boundaries of string world sheet. For partonic 2-surfaces the simplest option is that the edges are geodesic lines and thus have shortest possible length. This would bring in metric so that the idea about TGD as almost topological QFT would be realized.

One can distinguish between two cases: single polygon or several polygons.

1. One has maximal entanglement between fundamental fermions, when the vertices define single polygon. One can however have several polygons for a given set of vertices and in this case the coherence is reduced. Minimal correlations correspond to maximal number of 3-gons and minimal number of 4-gons and 5-gons.

2. For large $h_{eff} = n \times h$ the partonic 2-surfaces can have macroscopic and even astrophysical size and one can consider assigning many-fermion states with them. For instance, anyonic states could be interpreted in this manner. In this case it would be natural to consider various decompositions of the state to polygons representing entangled fermions.

The definition of symplectic invariant depends on whether one has single polygon or several polygons.

1. In the case that there are several polygons not containing polygons inside them (if this the case,then the complement of polygon must satisfy the condition) one can uniquely identify the interior of each polygon and assign a flux with it. Non-integrable phase factor is well-defined now. If there is only single polygon then also the complement of polygon could define the flux. Polygon and its complement define fluxes Φ and $\Phi_{tot} - \Phi$.

2. If partonic 2-surface carries monopole Kähler charge Φ_{tot} is essentially $n\pi$, where n is magnetic monopole flux through the partonic 2-surface. This is half integer - not integer: this is key feature of TGD and forces the coupling of Kähler gauge potential to the spinors leading to the quantum number spectrum of standard model. The exponent can be equal to -1 for half-odd integer.

   This problem disappears if both throats of the wormhole contact connecting the space-time sheets with Minkowski signature give their contribution so that two minus-signs give one plus sign. Elementary particles necessarily consist of wormhole contacts through which monopole flux flows and runs along second space-time sheet to another contact and returns along second space-time sheet so that closed monopole flux tube is obtained. The function of the flux must be single valued. This demands that it must reduce to the cosine of the integer multiple of the flux and identifiable as as the real part of the integer power of magnetic flux through the polygon.

The number theoretically deepest point is geometrically completely trivial.

1. Only $n > 2$-gons are non-degenerate and 3-, 4- and 5-gons are prime polygons in the sense that they cannot be sliced to lower polygons. Already 6-gon decomposes to 2 triangles.

2. One can wonder whether the appearance of 3 prime polygons might relate to family replication phenomenon for which TGD suggests an explanation in terms of genus of the partonic 2-surface [K9]. This does not seem to be the case. There is however other three special integers: namely 0,1, and 2.

The connection with family replication phenomenon could be following. When the number of handles at the parton surface exceeds 2, the system forms entangled/bound states describable in terms of polygons with handles at vertices. This would be kind of phase transition. Fundamental fermion families with handle number 0,1,2 would be analogous to integers 0,1,2 and the anyonic many-handle states with NE would be analogous to partitions of integers $n > 2$ represented by the prime polygons. They would correspond to the emergence of p-adic cognition. One could not assign NE and cognition with elementary particles but only to more complex objects such as anyonic states associated with large partonic 2-surfaces (perhaps large because they have large Planck constant $h_{eff} = n \times h$) [K31].

11.2.4 Integers $(3, 4, 5)$ as "additive primes" for integers $n \geq 3$: a connection with arithmetic consciousness

The above observations encourage a more detailed study of the decomposition of polygons to smaller polygons as a geometric representation for the partition of integers to a sum of smaller integers. The idea about integers $(3, 4, 5)$ as "additive primes" represented by prime polygons is especially attractive. This leads to a conjecture about NE associated with polygons as quantum correlates of arithmetic consciousness.

1. Motivations

The key idea is to look whether the notion of divisibility and primeness could have practical value in additive arithmetics. 1 is the only prime for addition in general case. $n = 1 + 1 + ...$ is analogous to p^n and all integers are "additive powers" of 1.

What happens if one considers integers $n \geq 3$? The basic motivation is that $n \geq 3$ is represented as a non-degenerate n-gon for $n \geq 3$. Therefore geometric representation of these primes is used in the following. One cannot split triangles from 4-gon and 5-gon. But already for 6-gon one can and obtains 2 triangles. Thus $\{3, 4, 5\}$ would be the additive primes for $n \geq 3$ represented as prime polygons.

The n-gons with $n \in \{3, 4, 5\}$ appear as faces of the Platonic solids! The inclusions of von Neumann algebras known as hyperfinite factors of type II_1 central in TGDs correspond to quantum phases $exp(\pi/n)$ $n = 3, 4, 5....$ Platonic solids correspond to particular finite subgroups of 3-D rotation group, which are in one-one correspondence with simply laced Lie-groups (ADE). There is also a direct connection with the classification of $\mathcal{N} = 2$ super-conformal theories, which seem to be relevant for TGD.

I cannot resist the temptation to mention also a personal reminiscence about a long lasting altered state of consciousness about 3 decades ago. I called it Great Experience and it boosted among other things serious work in order to understand consciousness in terms of quantum physics. One of the mathematical visions was that number 3 is in some sense fundamental for physics and mathematics. I also precognized infinite primes and much later indeed discovered them. I have repeatedly returned to the precognition about number 3 but found no really convincing reason for its unique role although it pops up again and again in physics and mathematics: 3 particle families, 3 colors for quarks, 3 spatial dimensions, 3 quaternionic imaginaryunits, triality for octonions, to say nothing about the role of trinity in mystics and religions. The following provides the first argument for the special role of number 3 that I can take seriously.

2. Partition of integer to additive primes

The problem is to find a partition of an integer to additive primes 3, 4, 5. The problem can be solved using a representation in terms of $n > 2$-gons as a geometrical visualization. Some general aspects of the representation.

1. The detailed shape of n-gons in the geometric representation of partitions does not matter: they just represent geometrically a partition of integer to a sum. The partition can be regarded as a dynamical process. n-gons splits to smaller n-gons producing a representation for a partition $n = \sum_i n_i$. What this means is easiest to grasp by imagining how polygon can be decomposed to smaller ones. Interestingly, the decompositions of polytopes to smaller ones - triangulations - appear also in Grassmannian twistor approach to $\mathcal{N} = 4$ super Yang Mills theory.

2. For a given partition the decomposition to n-gons is not unique. For instance, integer 12 can be represented by 3 4-gons or 4 3-gons. Integers $n \in \{3,4,5\}$ are special and partitions to these n-gons are in some sense maximal leading to a maximal decoherence as quantum physicist might say.

   The partitions are not unique and there is large number of partitions involving 3-gons,4-gons, 5-gons. The reason is that one can split from n-gons any n_1-gon with $n_1 < n$ except for $n = 3,4,5$.

3. The daydream of non-mathematician not knowing that everything has been very probably done for aeons ago is that one could chose n_1 to be indivisible by 4 and 5, n_2 indivisible by 3 and 5 and n_3 indivisible by 3 and 4 so that one might even hope for having a unique partition. For instance, double modding by 4 and 5 would reduce to double modding of $n_1 \times 3$ giving a non-vanishing result, and one might hope that n_1, n_2 and n_3 could be determined from the double modded values of n_i uniquely. Note that for $n_i \in \{1,2\}$ the number $n = 24 = 2 \times 3 + 2 \times 4 + 2 \times 5$ playing key role in string model related mathematics is the largest integer having this kind of representation. One should numerically check whether any general orbit characterized by the above formulas contains a point satisfying the additional number theoretic conditions.

   Therefore the task is to find partitions satisfying these indivisibility conditions. It is however reasonable to consider first general partitions.

4. By linearity the task of finding general partitions (forgetting divisibility conditions) is analogous to that of finding of solutions of non-homogenous linear equations. Suppose that one has found a partition

$$n = n_1 \times 3 + n_2 \times 4 + n_3 \times 5 \leftrightarrow (n_1, n_2, n_3) \ . \tag{11.1}$$

   This serves as the analog for the special solution of non-homogenous equation. One obtains a general solutions of equation as the sum $(n_1 + k_1, n_2 + k_1, n_3 + k_3)$ of the special solution and general solution of homogenous equation

$$k_1 \times 3 + k_2 \times 4 + k_3 \times 5 = 0 \ . \tag{11.2}$$

   This is equation of plane in N^3 - 3-D integer lattice.

   Using $4 = 3 + 1$ and $5 = 3 + 2$ this gives equations

$$k_2 + 2 \times k_3 = 3 \times m \ , \quad k_1 - k_3 + 4 \times m = 0 \ , \quad m = 0,1,2,... \tag{11.3}$$

5. There is periodicity of $3 \times 4 \times 5 = 60$. If (k_1, k_2, k_3, m) is allowed deformation, one obtains a new one with same divisibility properties as the original one as $(k_1 + 60, k_2 - 120, k_3 + 60, m)$. If one does not require divisibility properties for all solutions, one obtains much larger set of solutions. For instance $(k_1, k_2, k_3) = m \times (1, -2, 1)$ defines a line in the plane containing the solutions. Also other elementary moves than (1,-2,1) are possible.

One can identify very simple partitions deserving to be called standard partitions and involve mostly triangles and minimal number of 4- and 5-gons. The physical interpretation is that the coherence is minimal for them since mostly the quantum coherent negentropically entangled units are minimal triangles.

1. One starts from n vertices and constructs n-gon. For number theoretic purposes the shape does not matter and the polygon can be chosen to be convex. One slices from it 3-gons one by one so that eventually one is left with $k \equiv n \ mod \ 3 == 0, 1$ or 2 vertices. For $k = 0$ no further operations are needed. For $k = 1$ *resp.* $k = 2$ one combines one of the triangles and

edge associated with 1 *resp.* 2 vertices to 4-gon *resp.* 5-gon and is done. The outcome is one of the partitions

$$n = n_1 \times 3 \ , \quad n = n_1 \times 3 + 4, n = n_1 \times 3 + 5 \tag{11.4}$$

These partitions are very simple, and one can easily calculate similar partitions for products and powers. It is easy to write a computer program for the products and powers of integers in terms of these partitions.

2. There is however a uniqueness problem. If n_1 is divisible by 4 or 5 - $n_1 = 4 \times m_1$ or $n_1 = 5 \times m_1$ - one can interpret $n_1 \times 3$ as a collection of m_1 4-gons or 5-gons. Thus the geometric representation of the partition is not unique. Similar uniqueness condition must apply to n_2 and n_3 and is trivially true in above partitions.

   To overcome this problem one can pose a further requirement. If one wants n_1 to be indivisible by 4 and 5 one can transform 2 or or 4 triangles and existing 4-gon or 5-gon or 3 or 6 triangles to 4-gons and 5-gons.

   (a) Suppose $n = n_1 \times 3 + 4$. If n_1 divisible by 4 *resp.* 5 or both, $n_1 - 2$ is not and 4-gon and 2 3-gons can be transformed to 2 5-gons: $(n_1, 1, 0) \to (n_1 - 2, 0, 2)$. If $n_1 - 2$ is divisible by 5, $n_1 - 3$ is not divisible by either 4 or 5 and 3 triangles can be transformed to 4-gon and 5-gon: $(n_1, 1, 0) \to (n_1 - 3, 2, 1)$.

   (b) Suppose $n = n_1 \times 3 + 5$. If n_1 divisible by 4 *resp.* 5 or both, $n_1 - 1$ is not and triangle and 5-gon can be transformed to 2 4-gons: $(n_1, 0, 1) \to (n_1 - 1, 2, 0)$. If $n_1 - 1$ is divisible by 4 or 5, $n_1 - 3$ is not and 3 triangles and 5-gon can be transformed to 2 5-gons and 4-gon: $(n_1, 0, 1) \to (n_1 - 3, 1, 2)$.

   (c) For $n = n_1 \times 3$ divisible by 4 or 5 or both one can remove only $m \times 3$ triangles, $m \in \{1, 2\}$ since only in these case the resulting $m \times 3$ (9 or 18) vertices can partitioned to a union of 4-gon and 5-gon or of 2 4-gons and 2 5-gons: $(n_1, 0, 0) \to (n_1 - 3, 1, 1)$ or $(n_1, 0, 0) \to (n_1 - 6, 2, 2)$.

   These transformations seem to be the minimal transformations allowing to achieve indivisibility by starting from the partition with maximum number of triangles and minimal coherence.

Some further remarks about the partitions satisfying the divisibility conditions are in order.

1. The multiplication of n with partition (n_1, n_2, n_3) satisfying indivisibility conditions by an integer m not divisible by $k \in \{3, 4, 5\}$ gives integer with partition $m \times (n_1, n_2, n_3)$. Note also that if n is not divisible by $k \in \{3, 4, 5\}$ the powers of n, n^k has partition $n^{k-1} \times (n_1, n_2, n_3)$ and this could help to solve Diophantine equations.

2. Concerning the uniqueness of the partition satisfying the indivisibility conditions, the answer is negative. $8 = 3 + 5 = 4 + 4$ is the simplest counter example. Also the m-multiples of 8 such that m is indivisible by 2,3,4,5 serve as counter examples. 60-periodicity implies that for sufficiently large values of n the indivisibility conditions do not fix the partition uniquely. (n_1, n_2, n_3) can be replaced with $(n_1 + 60 + n_2 - 120, n_3 + 60)$ without affecting divisibility properties.

3. Intriguing observations related to 60-periodicity

60-periodicity seems to have deep connections with both music consciousness and genetic code if the TGD inspired model of genetic code is taken seriously code [K33] [L3].

1. The TGD inspired model for musical harmony and genetic involves icosahedron with 20 triangular faces and tetrahedron with 4 triangular faces. The 12 vertices of icosahedron correspond to the 12 notes. The model leads to the number 60. One can say that there are 60 +4 DNA codons and each 20 codon group is 60=20+20+20 corresponds to a subset

of aminoacids and 20 DNAs assignable to the triangles of icosahedron and representing also 3-chords of the associated harmony. The remaining 4 DNAs are associated with tetrahedron.

Geometrically the identification of harmonies is reduced to the construction of Hamiltonian cycles - closed isometrically non-equivalent non-self-intersecting paths at icosahedron going through all 12 vertices. The symmetries of the Hamiltonian cycles defined by subgroups of the icosahedral isometry group provide a classification of harmonies and suggest that also genetic code carries additional information assignable to what I call bio-harmony perhaps related to the expression of emotions - even at the level of biomolecules - in terms of "music" defined as sequences 3-chords realized in terms of triplets of dark photons (or notes) in 1-1 correspondence with DNA codons in given harmony.

2. Also the structure of time units and angle units involves number 60. Hour consists of 60 minutes, which consists of 60 seconds. Could this accident somehow reflect fundamental aspects of cognition? Could we be performing sub-conscious additive arithmetics using partitions of n-gons? Could it be possible to "see" the partitions if they correspond to NE?

4. Could additive primes be useful in Diophantine mathematics?

The natural question is whether it could be number theoretically practical to use "additive primes" $\{3, 4, 5\}$ in the construction of natural numbers $n \geq 3$ rather than number 1 and successor axiom. This might even provide a practical tool for solving Diophantine equations (it might well be that mathematicians have long ago discovered the additive primes).

The most famous Diophantine equation is $x^n + y^n = z^n$ and Fermat's theorem - proved by Wiles - states that for $n > 2$ it has no solutions. Non-mathematician can naïvely ask whether the proposed partition to additive primes could provide an elementary proof for Fermat's theorem and continue to test the patience of a real mathematician by wondering whether the parition for a sum of powers $n > 2$ could be always different from that for single power $n > 2$ perhaps because of some other constraints on the integers involved?

5. Could one identify quantum physical correlates for arithmetic consciousness?

Even animals and idiot savants can do arithmetics. How this is possible? Could one imagine physical correlates for arithmetic consciousness for which product and addition are the fundamental aspects? Is elementary arithmetic cognition universal and analogous to direct sensory experience. Could it reduce at quantum level to a kind of quantum measurement process quite generally giving rise to mental images as outcomes of quantum measurement by repeated state function reduction lasting as long as the corresponding sub-self (mental image) lives?

Consider a partition of integer to a product of primes first. I have proposed a general model for how partition of integer to primes could be experienced directly [L11]. For negentropically entangled state with maximal possible negentropy having entanglement probabilities $p_i = 1/N$, the negentropic primes are factors of N and they could be directly "seen" as negentropic p-adic factors in the adelic decomposition (reals and extensions of various p-adic number fields defined by extension of rationals defined the factors of adele and space-time surfaces as preferred extremals of Kähler action decompose to real and p-adic sectors).

What about additive arithmetics?

1. The physical motivation for n-gons is provided symplectic QFT [K8, K37], which is one aspect of TGD forced by super symplectic conformal invariance having structure of conformal symmetry. Symplectic QFT would be analogous to conformal QFT. The key challenge is to identify symplectic invariants on which the positive and negative energy parts of zero energy states can depend. The magnetic flux through a given area of 2-surface is key invariant of this kind. String world sheet and partonic 2-surfaces are possible identifications for the surface containing the polygon.

   Both the Kähler form associated with the light-cone boundary, which is metrically sphere with constant radius r_M (defining light-like radial coordinate) and the induced Kähler form of CP_2 define these kind of fluxes.

2. One can assign fluxes with string world sheets. In this case one has analog of magnetic flux but over a surface with metric signature (1,-1). Fluxes can be also assigned as magnetic fluxes

with partonic 2-surfaces at which fundamental fermions can be said to reside. n fermions defining the vertices at partonic 2-surface define naturally an n-gon or several of them. The interpretation would be as Abelian Wilson loop or equvalently non-integrable phase factor.

3. The polygons are not completely unique but this reflect the possibility of several physical states. n-gon could correspond to NE. The imaginary exponent of Kähler magnetic flux Φ through n-gon is symplectic invariant defining a non-integrable phase factor and defines a multiplicative factor of wave function. When the state decomposes to several polygons, one can uniquely identify the interior of the polygon and thus also the non-integrable phase factor.

   There is however non-uniqueness, when one has only single n-gon since also the complement of n-gon at partonic 2-surface containing now now polygons defines n-gon and the corresponding flux is $\Phi_{tot} - \Phi$. The flux Φ_{tot} is quantized and equal to the integer valued magnetic charge times 2π. The total flux disappears in the imaginary exponent and the non-integrable phase factor for the complementary polygon reduces to complex conjugate of that for polygon. Uniqueness allows only the cosine for an integer multiple of the flux.

   The non-integrable phase factor assignable to fermionic polygon would give rise to a correlation between fermions in zero modes invariant under symplectic group. The correlations defined by the n-gons at partonic 2-surfaces would be analogous to that in momentum space implied by the momentum conservation forcing the momenta to form a closed polygon but having totally different origin.

Could it be that the wave functions representing collections of n-gons representing partition of integer to a sum could be experienced directly by people capable of perplexing mathematical feats. The partition to a sum would correspond to a geometric partition of polygon representing partition of positive integer $n \geq 3$ to a sum of integers. Quantum physically it would correspond to NE as a representation of integer.

This might explain number theoretic miracles related to addition of integers in terms of direct "seeing". The arithmetic feats could be dynamical quantum processes in which polygons would decompose to smaller polygons, which would be directly "seen". This would require at least two representations: the original polygon and the decomposed polygon resulting in the state function reduction to the opposite boundary of CD. An ensemble of arithmetic sub-selves would seem to be needed. NMP does not seem to favour this kind of partition since negentropy is reduced but if its time reversal occurs in geometric time direction opposite to that of self it might look like partition for the self having sub-self as mental image.

12 p-Adicizable discrete variants of classical Lie groups and coset spaces in TGD framework

In TGD framework p-adicization and adelization are carried out at all levels of geometry: embedding space, space-time and WCW. Adelization at the level of state spaces requires that it is common from all sectors of the adele and has as coefficient field an extension of rationals allowing both real and p-adic interpretations: the sectors of adele give only different views about the same quantum state.

In the sequel the recent view about the p-adic variants of embedding space, space-time and WCW is discussed. The notion of finite measurement resolution reducing to number theoretic existence in p-adic sense is the fundamental notion. p-Adic geometries replace discrete points of discretization with p-adic analogs of monads of Leibniz making possible to construct differential calculus and formulate p-adic variants of field equations allowing to construct p-adic cognitive representations for real space-time surfaces.

This leads to a beautiful construction for the hierarchy of p-adic variants of embedding space inducing in turn the construction of p-adic variants of space-time surfaces. Number theoretical existence reduces to conditions demanding that all ordinary (hyperbolic) phases assignable to (hyperbolic) angles are expressible in terms of roots of unity (roots of e).

For $SU(2)$ one obtains as a special case Platonic solids and regular polygons as preferred p-adic geometries assignable also to the inclusions of hyperfinite factors [K44, K15]. Platonic solids represent idealized geometric objects ofthe p-adic world serving as a correlate for cognition as contrast to the geometric objects of the sensory world relying on real continuum.

In the case of causal diamonds (CDs) - the construction leads to the discrete variants of Lorentz group $SO(1,3)$ and hyperbolic spaces $SO(1,3)/SO(3)$. The construction gives not only the p-adicizable discrete subgroups of $SU(2)$ and $SU(3)$ but applies iteratively for all classical Lie groups meaning that the counterparts of Platonic solids are countered also for their p-adic coset spaces. Even the p-adic variants of WCW might be constructed if the general recipe for the construction of finite-dimensional symplectic groups applies also to the symplectic group assignable to $\Delta CD \times CP_2$.

The emergence of Platonic solids is very remarkable also from the point of view of TGD inspired theory of consciousness and quantum biology. For a couple of years ago I developed a model of music harmony [K33] [L3] relying on the geometries of icosahedron and tetrahedron. The basic observation is that 12-note scale can be represented as a closed curve connecting nearest number points (Hamiltonian cycle) at icosahedron going through all 12 vertices without self intersections. Icosahedron has also 20 triangles as faces. The idea is that the faces represent 3-chords for a given harmony characterized by Hamiltonian cycle. Also the interpretation terms of 20 amino-acids identifiable and genetic code with 3-chords identifiable as DNA codons consisting of three letters is highly suggestive.

One ends up with a model of music harmony predicting correctly the numbers of DNA codons coding for a given amino-acid. This however requires the inclusion of also tetrahedron. Why icosahedron should relate to music experience and genetic code? Icosahedral geometry and its dodecahedral dual as well as tetrahedral geometry appear frequently in molecular biology but its appearance as a preferred p-adic geometry is what provides an intuitive justification for the model of genetic code. Music experience involves both emotion and cognition. Musical notes could code for the points of p-adic geometries of the cognitive world. The model of harmony in fact generalizes. One can assign Hamiltonian cycles to any graph in any dimension and assign chords and harmonies with them. Hence one can ask whether music experience could be a form of p-adic geometric cognition in much more general sense.

The geometries of biomolecules brings strongly in mind the geometry p-adic space-time sheets. p-Adic space-time sheets can be regarded as collections of p-adic monad like objects at algebraic space-time points common to real and p-adic space-time sheets. Monad corresponds to p-adic units with norm smaller than unit. The collections of algebraic points defining the positions of monads and also intersections with real space-time sheets are highly symmetric and determined by the discrete p-adicizable subgroups of Lorentz group and color group. When the subgroup of the rotation group is finite one obtains polygons and Platonic solids. Bio-molecules typically consists of this kind of structures - such as regular hexagons and pentagons - and could be seen as cognitive representations of these geometries often called sacred! I have proposed this idea long time ago and the discovery of the recipe for the construction of p-adic geometries gave a justification for this idea.

12.1 p-Adic variants of causal diamonds

To construct p-adic variants of space-time surfaces one must construct p-adic variants of the embedding space. The assumption that the p-adic geometry for the embedding space induces p-adic geometry for sub-manifolds implies a huge simplification in the definition of p-adic variants of preferred extremals. The natural guess is that real and p-adic space-time surfaces gave algebraic points as common: so that the first challenge is to pick the algebraic points of the real space-time surface. To define p-adic space-time surface one needs field equations and the notion of p-adic continuum and by assigning to each algebraic point a p-adic continuum to make it monad, one can solve p-adic field equations inside these monads.

The idea of finite measurement resolution suggests that the solutions of p-adic field equations inside monads are arbitrary. Whether this is consistent with the idea that same solutions of field equations can be interpreted either p-adically or in real sense is not quite clear. This would be guaranteed if the p-adic solution has same formal representation as the real solution in the vicinity of given discrete point - say in terms of polynomials with rational coefficients and coordinate variables which vanish for the algebraic point.

Real and p-adic space-time surfaces would intersect at points common to all number fields for given adele: cognition and sensory worlds intersect not only at the level of WCW but also at the level of space-time. I had already considered giving up the latter assumption but it seems to be necessary at least for string world sheets and partonic 2-surfaces if not for entire space-time surfaces.

12.1.1 General recipe

The recipe would be following.

1. One starts from a discrete variant of $CD \times CP_2$ defined by an appropriate discrete symmetry groups and their subgroups using coset space construction. This discretization consists of points in finite-dimensional extension of p-adics induced by an extension of rationals. These points are assumed to be in the intersection of reality and p-adicities at space-time level - that is common for real and p-adic space-time surfaces. Cognitive representations in the real world are thus discrete and induced by the intersection. This is the original idea which I was ready to give up as the vision about discretization at WCW level allowing to solve all problems related to symmetries emerged. At space-time level the p-adic discretization reduces symmetry groups to their discrete subgroups: cognitive representations unavoidably break the symmetries. What is important the distance between discrete p-adic points labelling monads is naturally their real distance. This fixes metrically real-p-adic/sensory-cognitive correspondence.

2. One replaces each point of this discrete variant $CD \times CP_2$ with p-adic continuum defined by an algebraic extension of p-adics for the adele considered so that differentiation and therefore also p-adic field equations make sense. The continuum for given discrete point of $CD_d \times CP_{2,d}$ defines kind of Leibnizian monad representing field equations p-adically. The solution decomposes to p-adically differentiable pieces and the global solution of field equations makes sense since it can be interpreted in terms of pseudo-constants. p-Adicization means discretization but with discrete points replaced with p-adic monads preserving also the information about local behavior. The loss of well-ordering inside p-adic monad reflects its loss due to the finiteness of measurement resolution.

3. The distances between monads correspond to their distances for real variant of $CD \times CP_2$. Are there natural restrictions on the p-adic sizes of monads? Since p-adic units are in question that size in suitable units is $p^{-N} < 1$. It would look natural that the p-adic size of the is smaller than the distance to the nearest monad. The denser the discretization is, the larger the value of N would be. The size of the monad decreases at least like $1/p$ and for large primes assignable to elementary particles ($M_{127} = 2^{127} - 1$) is rather small. The discretizations of the subgroups share the properties of the group invariant geometry of groups so that they are to form a regular lattice like structure with constant distance to nearest neighbors. At the embedding level therefore p-adic geometries are extremely symmetric. At the level of space-time geometries only a subset of algebraic points is picked and the symmetry tends to be lost.

12.1.2 CD degrees of freedom

Consider first CD degrees of freedom.

1. For M^4 one has 4 linear coordinates. Should one p-adicize these or should one discretize CDs defined as intersections of future and past directed light-cones and strongly suggested by ZEO. CD seems to represent the more natural option. The construction of a given CD suggests that one should replace the usual representation of manifold as a union of overlapping regions with intersection of two light-cones with coordinates related in the intersection as in the case of ordinary manifold: $\cup \to \cap$.

2. For a given light-cone one must introduce light-cone proper time a, hyperbolic angle η and two angle coordinates (θ, ϕ). Light-cone proper time a is Lorentz invariant and corresponds naturally to an ordinary p-adic number of more generally to a p-adic number in algebraic extension which does not involve phases.

The two angle coordinates (θ, ϕ) parameterizing S^2 can be represented in terms of phases and discretized. The hyperbolic coordinate can be also discretized since e^p exists p-adically, and one obtains a finite-dimensional extension of p-adic numbers by adding roots of e and its powers. e is completely exceptional in that it is p-adically an algebraic number.

3. This procedure gives a discretization in angle coordinates. By replacing each discrete value of angle by p-adic continuum one obtains also now the monad structure. The replacement with continuum means the replacement

$$U_{m,n} \equiv exp(i2\pi m/n) \rightarrow U_{m,n} \times exp(i\phi) \ , \tag{12.1}$$

where ϕ is p-adic number with norm $p^{-N} < 1$ It can also belong to an algebraic extension of p-adic numbers. Building the monad is like replacing in finite measurement the representative point of measurement resolution interval with the entire interval. By finite measurement resolution one cannot fix the order inside the interval. Note that one obtains a hierarchy of subgroups depending on the upper bound p^{-n} for the modulus. For $p \ mod \ 4 = 1$ imaginary unit exist as ordinary p-adic number and for $p \ mod \ 4 = 3$ in an extension including $\sqrt{-1}$.

4. For the hyperbolic angle one has

$$E_{m,n} \equiv exp(m/n) \rightarrow E_{m,n} \times exp(\eta) \tag{12.2}$$

with the ordinary p-adic number η having norm $p^{-N} < 1$. Lorentz symmetry is broken to a discrete subgroup: this could be interpreted in terms of finite cognitive resolution. Since e^p is p-adic number also hyperbolic angle has finite number of values and one has compactness in well-defined sense although in real context one has non-compactness.

In cosmology this discretization means quantization of redshift and thus recession velocities. A concise manner to express the discretization to say that the cosmic time constant hyperboloids are discrete variants of Lobatchevski spaces $SO(3,1)/SO(3)$. The spaces appear naturally in TGD inspired cosmology.

5. The coordinate transformation relating the coordinates in the two intersecting coordinate patches maps hyperbolic and ordinary phases to each other as such. Light-cone proper time coordinates are related in more complex manner. $a_+^2 = t^2 - r^2$ and $a_-^2 = (t-T)^2 - r^2$ are related by $a_+^2 - a_-^2 = 2tT - T^2 = 2a_+cosh(\eta)T - T^2$.

This leads to a problem unless one allows a_+ and a_- to belong to an algebraic extension containing the roots of e making possible to define hyperbolic angle. The coordinates $a_\pm$ can also belong to a larger extension of p-adic numbers. The expectation is that one obtains an infinite hierarchy of algebraic extensions of rationals involving besides the phases also other non-Abelian extension parameters. It would seem that the Abelian extension for phases and the extension for a must factorize somehow. Note also that the expression of a_+ in terms of a_- given by

$$a_+ = -cosh(\eta)T \pm \sqrt{sinh^2(\eta)T^2 + a_-^2} \ . \tag{12.3}$$

This expression makes sense p-adically for all values of a_- if one can expand the square root as a converning power series with respect to a_-. This is true if $a_-/sinh(\eta)T$ has p-adic norm smaller than 1.

6. What about the boundary of CD which corresponds to a coordinate singularity? It seems that this must be treated separately. The boundary has topology $S^2 \times R_+$ and S^2 can be p-adicized as already explained. The light-like radial coordinate $r = asinh(\eta)$ vanishes identically for finite values of $sinh(\eta)$. Should one regard r as ordinary p-adic number? Or should one think that entire light-one boundary corresponds to single point $r = 0$? The discretization of

r is in powers of a roots of e is very natural so that each power $E_{m,n}$ corresponds to a p-adic monad. If now powers $E_{m,n}$ are involved, one obtains just the monad at $r = 0$.

The construction of quantum TGD leads to the introduction of powers $exp(log(r/r_0)s)$, where s is zero of Riemann Zeta [K13]. These make sense p-adically if $u = log(r/r_0)$ has p-adic norm smaller than unity and s makes sense p-adically. The latter condition demanding that the zeros are algebraic numbers is quite strong.

12.2 Construction for $SU(2)$, $SU(3)$, and classical Lie groups

In the following the detailed construction for $SU(2)$, $SU(3)$, and classical Lie groups will be sketched.

12.2.1 Subgroups of $SU(2)$ having p-adic counterparts

In the case $U(1)$ the subgroups defined by roots of unity reduce to a finite group Z_n. What can one say about p-adicizable discrete subgroups of $SU(2)$?

1. To see what happens in the case of $SU(2)$ one can write $SU(2)$ element explicitly in quaternionic matrix representation

$$(\theta, n) \equiv cos(\theta)Id + sin(\theta)\sum_i n_i I_i \quad . \tag{12.4}$$

   Here Id is quaternionic real unit and I_i are quaternionic imaginary units. $n = (n_1, n_2, n_2)$ is a unit vector representable as $(cos(\phi), sin(\phi)cos(\psi), sin(\phi)sin(\psi))$. This representation exists p-adically if the phases $exp(i\theta)$, $exp(i(\phi)$ and $exp(i\psi)$ exist p-adically so that they must be roots of unity.

   The geometric interpretation is that n defines the direction of rotation axis and θ defines the rotation angle.

2. This representation is not the most general one in p-adic context. Suppose that one has two elements of this kind characterized by (θ_i, n_i)such that the rotation axes are different. From the multiplication table of quaternions one has for the product (θ_{12}, n_{12}) of these

$$cos(\theta_{12}) \quad = \quad cos(\theta_1)cos(\theta_2) - sin(\theta_1)sin(\theta_2)n_1 \cdot n_2 \quad . \tag{12.5}$$

   This makes sense p-adically if the inner product $cos(\chi) \equiv n_1 \cdot n_2$ corresponds to root of unity in the extension of rationals used. Therefore the angle between the rotation axes is number theoretically quantized in order that p-adicization works.

   One can solve θ_{12} from the above equation in real context but in the general case it does not correspond to $U_{m,n}$. This is not however a problem from p-adic point of view. The reduction to a root of unity is true only in some special cases. For $n_1 = n_2$ the group generated by the products reduces a discrete $Z_n \subset U(1)$ generated by a root of unity. If n_1 and n_2 are orthogonal the angle between rotation axes corresponds trivially to a root of unity. In this case one has the isometries of cube. For other Platonic solids the angles between rotation axes associated with various $U(1)$ subgroups generating the entire sub-group are fixed by their geometries. The rotation angles correspond to $n = 3$ for tetrahedron and icosahedron and $n = 5$ dodecahedron and for $n = 3$. There is also duality between cube and octahedron and icosahedron and dodecahedron.

3. Platonic solids can be geometrically seen as discretized variants of $SU(2)$ and it seems that they correspond to finite discrete subgroups of $SU(2)$ defining $SU(2)_d$. Platonic sub-groups appear in the hierarchy of Jones inclusions. The other finite subgroups of $SU(2)$ appearing in this hierarchy act on polygons of plane and being generated by Z_n and rotations around the axes of plane and would naturally correspond to discrete U(1) sub-groups of $SU(2)$ and

in a well-defined sense to a degenerate situation. By Mc-Kay correspondence all these groups correspond to ADE type Lie groups. These subgroups define finite discretizations of $SU(2)$ and S^2. p-Adicization would lead directly to the hierarchy of inclusions assigned also with the hierarchy of sub-algebras of super-symplectic algebra characterized by the hierarchy of Planck constants.

4. There are also p-adicizable discrete subgroups, which are infinite. By taking two rotations with angles which correspond to root of unity with rotation axes, whose mutual angle corresponds to root of unity one can generate an infinite discrete subgroup of $SU(2)$ existing in p-adic sense. More general discrete U(1) subgroups are obtained by taking n rotation axes with mutual angles corresponding to roots of unity and generating the subgroup from these. In case of Platonic solids this gives a finite subgroup.

12.2.2 Construction of p-adicizable discrete subgroups of CP_2

The construction of p-adic CP_2 proceeds along similar lines.

1. In the original ultra-naïve approach the local p-adic metric of CP_2 is obtained by a purely formal replacement of the ordinary metric of CP_2 with its p-adic counterpart and it defines the CP_2 contribution to induced metric. This makes sense since Kähler function is rational function and components of CP_2 metric and spinor connection are rational functions. This allows to formulate p-adic variants of field equations. This description is however only local. It says nothing about global aspects of CP_2 related to the introduction of algebraic extension of p-adic numbers.

   One should be able to realize the angle coordinates of CP_2 in a physically acceptable manner. The coordinates of CP_2 can be expressed by compactness in terms of trigonometric functions, which suggests a realization of them as phases for the roots of unity. The number of points depends on the Abelian extension of rationals inducing that of p-adics which is chosen. This gives however only discrete version of p-adic CP_2 serving as a kind of spine. Also the flesh replacing points with monads is needed.

2. A more profound approach constructs the algebraic variants of CP_2 as discrete versions of the coset space $CP_2 = SU(3)/U(2)$. One restricts the consideration to an algebraic subgroup of $SU(3)_d$ with elements, which are 3×3 matrices with components, which are algebraic numbers in the extension of rationals. Since they are expressible in terms of phases one can express them in terms of roots of unity. In the same manner one identifies $U(2)_d \subset SU(3)_d$. $CP_{2,d}$ is the coset space $SU(3)_D/U(2)_d$ of these. The representative of a given coset is a point in the coset and expressible in terms of roots of unity.

3. The construction of the p-adicizable subgroups of $SU(3)$ suggests a generalization. Since $SU(3)$ is 8-D and Cartan algebra is 2-D the coset space is 6-dimensional flag-manifold $F = SU(3)/U(1) \times U(1)$ with coset consisting of elements related by automorphism $g \equiv hgh^{-1}$. F defines the twistor space of CP_2 characerizing the choices for the quantization axes of color quantum numbers. The points of F should be expressible in terms of phase angles analogous to the angle defining rotation axis in the case of SU(2).

   In the case of $SU(2)$ n U(1) subgroups with specified rotation axes with p-adically existing mutual angles are considered. The construction as such generates only $SU(2)_d$ subgroup which can be trivially extended to $U(2)_d$. The challenge is to proceed further.

   Cartan decomposition of the Lie algebra (see `http://tinyurl.com/y7cjbm4c`) seems to provide a solution to the problem. In the case of $SU(3)$ it corresponds to the decomposition to U(2) sub-algebra and its complement. One could use the decomposition $G = KAK$ where K is maximal compact subgroup. A is exponentiation of the maximal Abelian subalgebra, which is 3-dimensional for CP_2. By Abelianity the p-adicization of A in terms of roots of unity simple. The image of A in G/K is totally geodesic sub-manifold. In the recent case one has $G/Ki = CP_2$ so that the image of A is geodesic sphere S^2. This decomposition implies the representation using roots of unity. The construction of discrete p-adicizable subgroups of $SU(n)$ for $n > 3$ would continue iteratively.

4. Since the construction starts from $SU(2)$, $U(1)$, and Abelian groups, and proceeds iteratively it seems that Platonic solids have counterparts for all classical Lie groups containing $SU(2)$. Also level p-adicizable discrete coset spaces have analogous of Platonic solids.

The results imply that $CD \times CP_2$ is replaced by a discrete set of p-adic monads at a given level of hierarchy corresponding to the finite cognitive resolution.

12.2.3 Generalization to other groups

The above argument demonstrates that p-adicization works iteratively for $SU(n)$ and thus for $U(n)$. For finite-dimensional symplectic group $Sp(n,R)$ the maximal compact sub-group is $U(n)$ so that KAK construction should work also now. $SO(n)$ can be regarded as subgroup of $SU(n)$ so that the p-adiced discretrized variants of maximal compact subgroups should be constructible and KAK give the groups. The inspection of the table of the Wikipedia article (see `http://tinyurl.com/j44639q`) encourages the conjecture that the construction of $SU(n)$ and $U(n)$ generalizes to all classical Lie groups.

This construction could simplify enormously also the p-adicization of WCW and the theory would discretize even in non-compact degrees of freedom. The non-zero modes of WCW correspond to the symplectic group for $\delta M^4 \times CP_2$, and one might hope that the p-adicization works also at the limit of infinite-dimensional symplectic group with $U(\infty)$ taking the role of K.

13 Some layman considerations related to the fundamentals of mathematics

I am not a mathematician and therefore should refrain from consideration of anything related to fundamentals of mathematics. In the discussions with Santeri Satama I could not avoid the temptation to break this rule. I however feel that I must confess my sins and in the following I will do this.

1. Gödel's problematics is shown to have a topological analog in real topology, which however disappears in p-adic topology which raises the question whether the replacement of the arithmetics of natural numbers with that of p-adic integers could allow to avoid Gödel's problematics.

2. Number theory looks from the point of view of TGD more fundamental than set theory and inspires the question whether the notion of algebraic number could emerge naturally from TGD. There are two ways to understand the emergence of algebraic numbers: the hierarchy of infinite primes in which ordinary primes are starting point and the arithmetics of Hilbert spaces with tensor product and direct sum replacing the usual arithmetic operations. Extensions of rationals give also rise to cognitive variants of n-D spaces.

3. The notion of empty set looks artificial from the point of view of physicist and a possible cure is to take arithmetics as a model. Natural numbers would be analogous to nonempty sets and integers would correspond to pairs of sets (A, B), $A \subset B$ or $B \subset A$ with equivalence $A, B) \equiv (A \cup C, B \cup C)$. Empty set would correspond to pairs (A, A). In quantum context the generalization of the notion of being member of set $a \in A$ suggests a generalization: being an element in set would generalize to being single particle state which in general is de-localized to the set. Subsets would correspond to many-particle states. The basic operation would be addition or removal of element represented in terms of oscillator operator. The order of elements of set does not matter: this would generalize to bosonic and fermionic many particle states and even braid statistics can be considered. In bosonic case one can have multiple points - kind of Bose-Einstein condensate.

4. One can also start from finite-D Hilbert space and identify set as the collection of labels for the states. In infinite-D case there are two cases corresponding to separable and non-separable Hilbert spaces. The condition that the norm of the state is finite without infinite normalization constants forces selection of de-localized discrete basis in the case of a continuous set like reals. This inspires the question whether the axiom of choice should be given

up. One possibility is that one can have only states localized to finite or at least discrete set of points which correspond points with coordinates in an extension of rationals.

13.1 Geometric analog for Gödel's problematics

Godel's problematics involves statements which cannot be proved to be true or false or are simultaneously true and false. This problematics has also a purely geometric analog in terms of set theoretic representation of Boolean algebras when real topology is used but not when p-adic topology is used.

The natural idea is that Boolean algebra is realized in terms of open sets such that the negation of statement corresponds to the complement of the set. In p-adic topologies open sets are simultaneously also closed and there are no boundaries: this makes them and - more generally Stone spaces - ideal for realizing Boolean algebra set theoretically. In real topology the complement of open set is closed and therefore not open and one has a problem.

Could one circumvent the problem somehow?

1. If one replaces open sets with their closures (the closure of open set includes also its boundary, which does not belong to the open set) and closed complements of open sets, the analog of Boolean algebra would consist of closed sets. Closure of an open set and the closure of its open complement - stament and its negation - share the common boundary. Statement and its negation would be simultaneously true at the boundary. This strange situation reminds of Russell's paradox but in geometric form.

2. If one replaces the closed complements of open sets with their open interiors, one has only open sets. Now the sphere would represent statement about which one cannot say whether it is true or false. This would look like Gödelian sentence but represented geometrically.

   This leads to an already familiar conclusion: p-adic topology is natural for the geometric correlates of cognition, in particular Boolean cognition. Real topology is natural for the geometric correlates of sensory experience.

3. Gödelian problematics is encountered already for arithmetics of natural numbers although naturals have no boundary in the discrete topology. Discrete topology does not however allow well-ordering of natural numbers crucial for the definition of natural number. In the induced real topology one can order them and can speak of boundaries of subsets of naturals. The ordering of natural numbers by size reflects the ordering of reals: it is very difficult to think about discrete without implicitly bringing in the continuum.

   For p-adic integers the induced topology is p-adic. Is Gödelian problematics is absent in p-adic Boolean logic in which set and its complement are both open and closed. If this view is correct, p-adic integers might replace naturals in the axiomatics of arithmetics. The new element would be that most p-adic integers are of infinite size in real sense. One has a natural division of them to cognitively representable ones finite also in real sense and non-representable ones infinite in real sense. Note however that rationals have periodic pinary expansion and can be represented as pairs of finite natural numbers.

In algebraic geometry Zariski topology in which closed sets correspond to algebraic surfaces of various dimensions, is natural. Open sets correspond to their complements and are of same dimension as the embedding space. Also now one encounters asymmetry. Could one say that algebraic surfaces characterize "representable" (="geometrically provable"?) statements as elements of Boolean algebra and their complements the non-representable ones? 4-D space-time (as possibly associative/co-associative) algebraic variety in 8-D octonionic space would be example of representable statement. Finite unions and intersections of algebraic surfaces would form the set of representable statements. This new-to-me notion of representability is somehow analogous to provability or demonstrability.

13.2 Number theory from quantum theory

Could one define or at least represent the notion of number using the notions of quantum physics? A natural starting point is hierarchy of extensions of rationals defining hierarchy of adeles. Could one obtain rationals and their extensions from simplest possible quantum theory in which one just constructs many particle states by adding or removing particles using creation and annihilation operators?

13.2.1 How to obtain rationals and their extensions?

Rationals and their extensions are fundamental in TGD. Can one have quantal construction for them?

1. One should construct rationals first. Suppose one starts from the notion of finite prime as something God-given. At the first step one constructs infinite primes as analogs for many-particle states in super-symmetric arithmetic quantum field theory [K36]. Ordinary primes label states of fermions and bosons. Infinite primes as the analogs of free many-particle states correspond to rationals in a natural manner.

2. One obtains also analogs of bound states which are mappable to irreducible polynomials, whose roots define algebraic numbers. This would give hierarchy of algebraic extensions of rationals. At higher levels of the hierarchy one obtains also analogs of prime polynomials with number of variables larger than 1. One might say that algebraic geometry has quantal representation. This might be very relevant for the physical representability of basic mathematical structures.

13.2.2 Arithmetics of Hilbert spaces

The notions of prime and divisibility and even basic arithmetics emerge also from the tensor product and direct sum for Hilbert spaces. Hilbert spaces with prime dimension do not decompose to tensor products of lower-dimensional Hilbert spaces. One can even perform a formal generalization of the dimension of Hilbert space so that it becomes rational and even algebraic number.

For some years ago I indeed played with this thought but at that time I did not have in mind reduction of number theory to the arithemetics of Hilbert spaces. If this really makes sense, numbers could be replaced by Hilbert spaces with product and sum identified as tensor product and direct sum!

Finite-dimensional Hilbert space represent the analogs of natural numbers. The analogs of integers could be defined as pairs (m,n) of Hilbert spaces with spaces (m,n) and $(m+r,n+r)$ identified (this space would have dimension $m-n$. This identification would hold true also at the level of states. Hilbert spaces with negative dimension would correspond to pairs with $(m-n)<0$: the canonical representives for m and $-m$ would be $(m,0)$ and $(0,m)$. Rationals can be defined as pairs (m,n) of Hilbert spaces with pairs (m,n) and (km,kn) identified. These identifications would give rise to kind of gauge conditions and canonical representatives for m and $1/m$ are $(m,1)$ and $(1,m)$.

What about Hilbert spaces for which the dimension is algebraic number? Algebraic numbers allow a description in terms of partial fractions and Stern-Brocot (S-B) tree (see `http://tinyurl.com/yb6ldekq` and `http://tinyurl.com/yc6hhboo`) containing given rational number once. S-B tree allows to see information about algebraic numbers as constructible by using an algorithm with finite number of steps, which is allowed if one accepts abstraction as basic aspect of cognition. Algebraic number could be seen as a periodic partial fraction defining an infinite path in S-B tree. Each node along this path would correspond to a rational having Hilbert space analog. Hilbert space with algebraic dimension would correspond to this kind of path in the space of Hilbert spaces with rational dimension. Transcendentals allow identification as non-pediodic partial fraction and could correspond to non-periodic paths so that also they could have Hilbert spaces counterparts.

13.2.3 How to obtain the analogs higher-D spaces?

Algebraic extensions of rationals allow cognitive realization of spaces with arbitrary dimension identified as algebraic dimension of extension of rationals.

1. One can obtain n-dimensional spaces (in algebraic sense) with integer valued coordinates from n-D extensions of rationals. Now the n-tuples defining numbers of extension and differing

by permutations are not equivalent so that one obtains n-D space rather than n-D space divided by permutation group S_n. This is enough at the level of cognitive representations and could explain why we are able to imagine spaces of arbitrary dimension although we cannot represent them cognitively.

2. One obtains also Galois group and orbits of set A of points of extension under Galois group as $G(A)$. One obtains also discrete coset spaces G/H and alike. These do not have any direct analog in the set theory. The hierarchy of Galois groups would bring in discrete group theory automatically. The basic machinery of quantum theory emerges elegantly from number theoretic vision.

3. In octonionic approach to quantum TGD one obtains also hierarchy of extensions of rationals since space-time surface correspond zero loci for RE or IM for octonionic polynomials obtained by algebraic continuation from real polynomials with coeffficients in extension of rationals [**?**].

13.3 Could quantum set theory make sense?

In the following my view point is that of quantum physicist fascinated by number theory and willing to reduce set theory to what could be called called quantum set theory. It would follow from physics as generalised number theory (adelic physics) and have ordinary set theory as classical correlate.

1. From the point of quantum physics set theory and the notion of number based on set theory look somewhat artificial constructs. Nonempty set is a natural concept but empty set and set having empty set as element used as basic building brick in the construction of natural numbers looks weird to me.

2. From TGD point of view it would seem that number theory plus some basic pieces of quantum theory might be more fundamental than set theory. Could set theory emerge as a classical correlate for quantum number theory already considered and could quantal set theory make sense?

13.3.1 Quantum set theory

What quantum set theory could mean? Suppose that number theory-quantum theory connection really works. What about set theory? Or perhaps its quantum counterpart having ordinary set theory as a classical correlate?

1. A purely quantal input to the notion of set would be replacement of points delocalized states in the set. A generic single particle quantum state as analog of element of set would not be localized to a single element of set. The condition that the state has finite norm implies in the case of continuous set like reals that one cannot have completely localized states. This would give quantal limitation to the axiom of choice. One can have any discrete basis of state functions in the set but one cannot pick up just one point since this state would have infinite norm.

   The idea about allowing only say rationals is not needed since there is infinite number of different choices of basis. Finite measurement resolution is however unvoidable. An alternative option is restriction of the domains of wave functions to a discrete set of points. This set can be chosen in very many ways and points with coordinates in extension of rationals are very natural and would define cognitive representation.

2. One can construct also the analogs of subsets as many-particle states. The basic operation would be addition/removal of a particle from quantum state represented by the action of creation/annihilation operator.

   Bosonic states would be invariant under permutations of single particle states just like set is the equivalence class for a collection of elements $(a_1, ..., a_n)$ such that any two permutations are equivalent. Quantum set theory would however bring in something new: the possibility of both bosonic and fermionic statistics. Permutation would change the state by phase factor -1. One would have fermionic and bosonic sets. For bosonic sets one could have multiplet elements ("Bose-Einstein condensation"): in the theory of surfaces this could allow multiple copies of the same surface. Even braid statistics is possible. The phase factor in permutation could be complex. Even non-commutative statistics can be considered.

Many particle states formed from particles, which are not identical are also possible and now the different particle types can be ordered. On obtains n-ples decomposing to ordered K-ple of n_i-ples, which are consist of identical particles and are quantum sets. One could talk about K-sets as a generalization of set as analogs of classical sets with K-colored elements. Group theory would enter into the picture via permutation groups and braid groups would bring in braid statistics. Braids strands would have K colors.

13.3.2 How to obtain classical set theory?

How could one obtain classical set theory?

1. Many-particle states represented algebraically are detected in lab as sets: this is quantum classical correspondence. This remains to me one of the really mysterious looking aspects in the interpretation of quantum field theory. For some reason it is usually not mentioned at all in popularizations. The reason is probably that popularization deals typically with wave mechanics but not quantum field theory unless it is about Higgs mechanism, which is the weakest part of quantum field theory!

2. From the point of quantum theory empty set would correspond to vacuum. It is not observable as such. Could the situation change in the presence of second state representing the environment? Could the fundamental sets be always *non-empty* and correspond to states with non-vanishing particle number. Natural numbers would correspond to eigenvalues of an observable telling the cardinality of set. Could representable sets be like natural numbers?

3. Usually integers are identified as pairs of natural numbers (m, n) such that integer corresponds to $m - n$. Could the set theoretic analog of integer be a pair (A, B) of sets such that A is subset of B or vice versa? Note that this does not allow pairs with disjoint members. (A, A) would correspond to empty set. This would give rise to sets (A, B) and their "antisets" (B, A) as analogs of positive and negative integers.

   One can argue that antisets are not physically realizable. Sets and antisets would have as analogs two quantizations in which the roles of oscillator operators and their hermitian conjugates are changed. The operators annihilating the ground state are called annilation operators. Only either of these realization is possible but not both simultaneously.

   In ZEO one can ask whether these two options correspond to positive and negative energy parts of zero energy states or to the states with state function reduction at either boundary of CD identified as correlates for conscious entities with opposite arrows of geometric time (generalized Zeno effect).

4. The cardinality of set, the number of elements in the set, could correspond to eigenvalue of observable measuring particle number. Many-particle states consisting of bosons or fermions would be analogs for sets since the ordering does not matter. Also braid statistics would be possible.

   What about cardinality as a p-adic integer? In p-adic context one can assign to integer m, integer $-m$ as $m \times (p-1) \times (1 + p + p^2 + ...)$. This is infinite as real integer but finite as p-adic integer. Could one say that the antiset of m-element as analog of negative integer has cardinality $-m = m(p-1)(1+p+p^2+..)$. This number does not have cognitive representation since it is not finite as real number but is cognizable.

   One could argue that negative numbers are cognizable but not cognitively representable as cardinality of set? This representation must be distinguished from cognitive representations as a point of embedding space with coordinates in extension of rationals. Could one say that antisets and empty set as its own antiset can be cognized but cannot be cognitively represented?

Nasty mathematician would ask whether I can really start from Hilbert space of state functions and deduce from this the underlying set. The elements of set itself should emerge from this as analogs of completely localized single particle states labelled by points of set. In the case of finite-dimensional Hilbert space this is trivial. The number of points in the set would be equal to the dimension of Hilbert space. In the case of infinite-D Hilbert space the set would have infinite number of points.

Here one has two views about infinite set. One has both separable (infinite-D in discrete sense: particle in box with discrete momentum spectrum) and non-separable (infinite-D in real sense: free particle with continuous momentum spectrum) Hilbert spaces. In the latter case the completely localized single particle states would be represented by delta functions divided by infinite normalization factors. They are routinely used in Dirac's bra-ket formalism but problems emerge in quantum field theory.

A possible solution is that one weakens the axiom of choice and accepts that only discrete points set (possibly finite) are cognitively representable and one has wave functions localized to discrete set of points. A stronger assumption is that these points have coordinates in extension of rationals so that one obtains number theoretical universality and adeles. This is TGD view and conforms also with the identification of hyper-finite factors of type II_1 as basic algebraic objects in TGD based quantum theory as opposed to wave mechanics (type I) and quantum field theory (type III). They are infinite-D but allow excellent approximation as finite-D objects.

This picture could relate to the notion of non-commutative geometry, where set emerges as spectrum of algebra: the points of spectrum label the ideals of the integer elements of algebra.

14 Abelian Class Field Theory And TGD

The context leading to the discovery of adeles (`http://tinyurl.com/64pgerm`) was so called Abelian class field theory. Typically the extension of rationals means that the ordinary primes decompose to the primes of the extension just like ordinary integers decompose to ordinary primes. Some primes can appear several times in the decomposition of ordinary non-square-free integers and similar phenomenon takes place for the integers of extension. If this takes place one says that the original prime is ramified. The simplest example is provided Gaussian integers $Q(i)$. All odd primes are unramified and primes $p \ mod \ 4 = 1$ they decompose as $p = (a + ib)(a - ib)$ whereas primes $p \ mos \ 4 = 3$ do not decompose at all. For $p = 2$ the decomposition is $2 = (1 + i)(1 - i) = -i(1 + i)^2 = i(1 - i)^2$ and is not unique $\{\pm 1, \pm i\}$ are the units of the extension. Hence $p = 2$ is ramified.

There goal of Abelian class field theory (see `http://tinyurl.com/y8aefmg2`) is to understand the complexities related to the factorization of primes of the original field. The existence of the isomorphism between ideles modulo rationals - briefly ideles - and maximal Abelian Galois Group of rationals (MAGG) is one of the great discoveries of Abelian class field theory. Also the maximal - necessarily Abelian - extension of finite field G_p has Galois group isomorphic to the ideles. The Galois group of $G_p(n)$ with p^n elements is actually the cyclic group Z_n. The isomorphism opens up the way to study the representations of Abelian Galois group and also those of the AGG. One can indeed see these representations as special kind of representations for which the commutator group of AGG is represented trivially playing a role analogous to that of gauge group.

This framework is extremely general. One can replace rationals with any algebraic extension of rationals and study the maximal Abelian extension or algebraic numbers as its extension. One can consider the maximal algebraic extension of finite fields consisting of union of all all finite fields associated with given prime and corresponding adele. One can study function fields defined by the rational functions on algebraic curve defined in finite field and its maximal extension to include Taylor series. The isomorphisms applies in al these cases. One ends up with the idea that one can represent maximal Abelian Galois group in function space of complex valued functions in $GL_e(A)$ right invariant under the action of $GL_e(Q)$. A denotes here adeles.

In the following I will introduce basic facts about adeles and ideles and then consider a possible realization of the number theoretical vision about quantum TGD as a Galois theory for the algebraic extensions of classical number fields with associativity defining the dynamics. This picture leads automatically to the adele defined by p-adic variants of quaternions and octonions, which can be defined by posing a suitable restriction consistent with the basic physical picture provide by TGD.

14.1 Adeles And Ideles

Adeles and ideles are structures obtained as products of real and p-adic number fields. The formula expressing the real norm of rational numbers as the product of inverses of its p-adic norms inspires the idea about a structure defined as produc of reals and various p-adic number fields.

Class field theory (`http://tinyurl.com/y8aefmg2`) studies Abelian extensions of global fields (classical number fields or functions on curves over finite fields), which by definition have Abelian Galois group acting as automorphisms. The basic result of class field theory is one-one correspondence between Abelian extensions and appropriate classes of ideals of the global field or open subgroups of the ideal class group of the field. For instance, Hilbert class field, which is maximal unramied extension of global field corresponds to a unique class of ideals of the number field. More precisely, reciprocity homomorphism generalizes the quadratic resiprocity for quadratic extensions of rationals. It maps the idele class group of the global field defined as the quotient of the ideles by the multiplicative group of the field - to the Galois group of the maximal Abelian extension of the global field. Each open subgroup of the idele class group of a global field is the image with respect to the norm map from the corresponding class field extension down to the global field.

The idea of number theoretic Langlands correspondence, [K19, A9, A8]. is that n-dimensional representations of Absolute Galois group correspond to infinite-D unitary representations of group $Gl_n(A)$. Obviously this correspondence is extremely general but might be highly relevant for TGD, where embedding space is replaced with Cartesian product of real embedding space and its p-adic variants - something which might be related to octonionic and quaternionic variants of adeles. It seems however that the TGD analogs for finite-D matrix groups are analogs of local gauge groups or Kac-Moody groups (in particular symplectic group of $\delta M^4_+ \times CP_2$) so that quite heavy generalization of already extremely abstract formalism is expected.

The following gives some more precise definitions for the basic notions.

1. Prime ideals of global field, say that of rationals, are defined as ideals which do not decompose to a product of ideals: this notion generalizes the notion of prime. For instance, for p-adic numbers integers vanishing mod p^n define an ideal and ideals can be multiplied. For Abelian extensions of a global field the prime ideals in general decompose to prime ideals of the extension, and the decompostion need not be unique: one speaks of ramification. One of the challenges of tjhe class field theory is to provide information about the ramification. Hilbert class field is define as the maximal unramified extension of global field.

2. The ring of integral adeles (see `http://tinyurl.com/64pgerm`) is defined as $A_Z = R \times \hat{Z}$, where $\hat{Z} = \prod_p Z_p$ is Cartesian product of rings of p-adic integers for all primes (prime ideals) p of assignable to the global field. Multiplication of element of A_Z by integer means multiplication in all factors so that the structure is like direct sum from the point of view of physicist.

3. The ring of rational adeles can be defined as the tensor product $A_Q = Q \otimes_Z A_Z$. $_Z$ means that in the multiplication by element of Z the factors of the integer can be distributed freely among the factors $\hat{Z}$. Using quantum physics language, the tensor product makes possible entanglement between Q and A_Z.

4. Another definition for rational adeles is as $R \times \prod'_p Q_p$: the rationals in tensor factor Q have been absorbed to p-adic number fields: given prime power in Q has been absorbed to corresponding Q_p. Here all but finite number of Q_p elements ar p-adic integers. Note that one can take out negative powers of p_i and if their number is not finite the resulting number vanishes.The multiplication by integer makes sense but the multiplication by a rational does not smake sense since all factors Q_p would be multiplied.

5. Ideles are defined as invertible adeles (`http://tinyurl.com/yc3yrcxx`Idele class group). The basic result of the class field theory is that the quotient of the multiplicative group of ideles by number field is homomorphic to the maximal Abelian Galois group!

14.2 Questions About Adeles, Ideles And Quantum TGD

The intriguing general result of class field theory (`http://tinyurl.com/y8aefmg2`) is that the the maximal Abelian extension for rationals is homomorphic with the multiplicative group of ideles.

This correspondence plays a key role in Langlands correspondence.

Does this mean that it is not absolutely necessary to introduce p-adic numbers? This is actually not so. The Galois group of the maximal abelian extension is rather complex objects (absolute Galois group, AGG, defines as the Galois group of algebraic numbers is even more complex!). The ring $\hat{Z}$ of adeles defining the group of ideles as its invertible elements homeomorphic to the Galois group of maximal Abelian extension is profinite group (http://tinyurl.com/y9d8vro7). This means that it is totally disconnected space as also p-adic integers and numbers are. What is intriguing that p-dic integers are however a continuous structure in the sense that differential calculus is possible. A concrete example is provided by 2-adic units consisting of bit sequences which can have literally infinite non-vanishing bits. This space is formally discrete but one can construct differential calculus since the situation is not democratic. The higher the pinary digit in the expansion is, the less significant it is, and p-adic norm approaching to zero expresses the reduction of the insignificance.

1. Could TGD based physics reduce to a representation theory for the Galois groups of quaternions and octonions?

Number theoretical vision about TGD raises questions about whether adeles and ideles could be helpful in the formulation of TGD. I have already earlier considered the idea that quantum TGD could reduce to a representation theory of appropriate Galois groups. I proceed to make questions.

1. Could real physics and various p-adic physics on one hand, and number theoretic physics based on maximal Abelian extension of rational octonions and quaternions on one hand, define equivalent formulations of physics?

2. Besides various p-adic physics all classical number fields (reals, complex numbers, quaternions, and octonions) are central in the number theoretical vision about TGD. The technical problem is that p-adic quaternions and octonions exist only as a ring unless one poses some additional conditions. Is it possible to pose such conditions so that one could define what might be called quaternionic and octonionic adeles and ideles?

   It will be found that this is the case: p-adic quaternions/octonions would be products of rational quaternions/octonions with a p-adic unit. This definition applies also to algebraic extensions of rationals and makes it possible to define the notion of derivative for corresponding adeles. Furthermore, the rational quaternions define non-commutative automorphisms of quaternions and rational octonions at least formally define a non-associative analog of group of octonionic automorphisms [K37, K43].

3. I have already earlier considered the idea about Galois group as the ultimate symmetry group of physics. The representations of Galois group of maximal Abelian extension (or even that for algebraic numbers) would define the quantum states. The representation space could be group algebra of the Galois group and in Abelian case equivalently the group algebra of ideles or adeles. One would have wave functions in the space of ideles.

   The Galois group of maximal Abelian extension would be the Cartan subgroup of the absolute Galois group of algebraic numbers associated with given extension of rationals and it would be natural to classify the quantum states by the corresponding quantum numbers (number theoretic observables).

   If octonionic and quaternionic (associative) adeles make sense, the associativity condition would reduce the analogs of wave functions to those at 4-dimensional associative sub-manifolds of octonionic adeles identifiable as space-time surfaces so that also space-time physics in various number fields would result as representations of Galois group in the maximal Abelian Galois group of rational octonions/quaternions. TGD would reduce to classical number theory! One can hope that WCW spinor fields assignable to the associative and co-associative space-time surfaces provide the adelic representations for super-conformal algebras replacing symmetries for point like objects.

   This of course involves huge challenges: one should find an adelic formulation for WCW in terms octonionic and quaternionic adeles, similar formulation for WCW spinor fields in terms of adelic induced spinor fields or their octonionic variants is needed. Also zero energy ontology, causal diamonds, light-like 3-surfaces at which the signature of the induced metric changes, space-like 3-surfaces and partonic 2-surfaces at the boundaries of CDs, $M^8 - H$ duality, possible representation of space-time surfaces in terms of of O_c-real analytic functions (O_c denotes for complexified octonions), etc. should be generalized to adelic framework.

4. Absolute Galois group is the Galois group of the maximal algebraic extension and as such a poorly defined concept. One can however consider the hierarchy of all finite-dimensional algebraic extensions (including non-Abelian ones) and maximal Abelian extensions associated with these and obtain in this manner a hierarchy of physics defined as representations of these Galois groups homomorphic with the corresponding idele groups.

5. In this approach the symmetries of the theory would have automatically adelic representations and one might hope about connection with Langlands program [K19], [K19, A9, A8].

2. Adelic variant of space-time dynamics and spinorial dynamics?

As an innocent novice I can continue to pose stupid questions. Now about adelic variant of the space-time dynamics based on the generalization of Kähler action discussed already earlier but without mentioning adeles ([K46]).

1. Could one think that adeles or ideles could extend reals in the formulation of the theory: note that reals are included as Cartesian factor to adeles. Could one speak about adelic space-time surfaces endowed with adelic coordinates? Could one formulate variational principle in terms of adeles so that exponent of action would be product of actions exponents associated with various factors with Neper number replaced by p for Z_p. The minimal interpretation would be that in adelic picture one collects under the same umbrella real physics and various p-adic physics.

2. Number theoretic vision suggests that 4: th/8: th Cartesian powers of adeles have interpretation as adelic variants of quaternions/ octonions. If so, one can ask whether adelic quaternions and octonions could have some number theoretical meaning. Adelic quaternions and octonions are not number fields without additional assumptions since the moduli squared for a p-adic analog of quaternion and octonion can vanish so that the inverse fails to exist at the light-cone boundary which is 17-dimensional for complexified octonions and 7-dimensional for complexified quaternions. The reason is that norm squared is difference $N(o_1) - N(o_2)$ for $o_1 \oplus io_2$. This allows to define differential calculus for Taylor series and one can consider even rational functions. Hence the restriction is not fatal.

   If one can pose a condition guaranteeing the existence of inverse for octonionic adel, one could define the multiplicative group of ideles for quaternions. For octonions one would obtain non-associative analog of the multiplicative group. If this kind of structures exist then four-dimensional associative/co-associative sub-manifolds in the space of non-associative ideles define associative/co-associative adeles in which ideles act. It is easy to find that octonionic ideles form 1-dimensional objects so that one must accept octonions with arbitrary real or p-adic components.

3. What about equations for space-time surfaces. Do field equations reduce to separate field equations for each factor? Can one pose as an additional condition the constraint that p-adic surfaces provide in some sense cognitive representations of real space-time surfaces: this idea is formulated more precisely in terms of p-adic manifold concept [K46] (see the appendix of the book). Or is this correspondence an outcome of evolution?

   Physical intuition would suggest that in most p-adic factors space-time surface corresponds to a point, or at least to a vacuum extremal. One can consider also the possibility that same algebraic equation describes the surface in various factors of the adele. Could this hold true in the intersection of real and p-adic worlds for which rationals appear in the polynomials defining the preferred extremals.

4. To define field equations one must have the notion of derivative. Derivative is an operation involving division and can be tricky since adeles are not number field. The above argument

suggests this is not actually a problem. Of course, if one can guarantee that the p-adic variants of octonions and quaternions are number fields, there are good hopes about well-defined derivative. Derivative as limiting value $df/dx = \lim(f(x+dx) - f(x))/dx$ for a function decomposing to Cartesian product of real function $f(x)$ and p-adic valued functions $f_p(x_p)$ would require that $f_p(x)$ is non-constant only for a finite number of primes: this is in accordance with the physical picture that only finite number of p-adic primes are active and define "cognitive representations" of real space-time surface. The second condition is that dx is proportional to product $dx \times \prod dx_p$ of differentials dx and dx_p, which are rational numbers. dx goes to xero as a real number but not p-adically for any of the primes involved. dx_p in turn goes to zero p-adically only for Q_p.

5. The idea about rationals as points common to all number fields is central in number theoretical vision. This vision is realized for adeles in the minimal sense that the action of rationals is well-defined in all Cartesian factors of the adeles. Number theoretical vision allows also to talk about common rational points of real and various p-adic space-time surfaces in preferred coordinate choices made possible by symmetries of the embedding space, and one ends up to the vision about life as something residing in the intersection of real and p-adic number fields. It is not clear whether and how adeles could allow to formulate this idea.

6. For adelic variants of embedding space spinors Cartesian product of real and p-adc variants of embedding spaces is mapped to their tensor product. This gives justification for the physical vision that various p-adic physics appear as tensor factors. Does this mean that the generalized induced spinors are infinite tensor products of real and various p-adic spinors and Clifford algebra generated by induced gamma matrices is obtained by tensor product construction? Does the generalization of massless Dirac equation reduce to a sum of d'Alembertians for the factors? Does each of them annihilate the appropriate spinor? If only finite number of Cartesian factors corresponds to a space-time surface which is not vacuum extremal vanishing induced Kähler form, Kähler Dirac equation is non-trivial only in finite number of adelic factors.

3. Objections leading to the identification of octonionic adeles and ideles

The basic idea is that appropriately defined invertible quaternionic/octonionic adeles can be regarded as elements of Galois group assignable to quaternions/octonions. The best manner to proceed is to invent objections against this idea.

1. The first objection is that p-adic quaternions and octonions do not make sense since p-adic variants of quaternions and octonions do not exist in general. The reason is that the p-adic norm squared $\sum x_i^2$ for p-adic variant of quaternion, octonion, or even complex number can vanish so that its inverse does not exist.

2. Second objection is that automorphisms of the ring of quaternions (octonions) in the maximal Abelian extension are products of transformations of the subgroup of $SO(3)$ (G_2) represented by matrices with elements in the extension and in the Galois group of the extension itself. Ideles separate out as 1-dimensional Cartesian factor from this group so that one does not obtain 4-field (8-fold) Cartesian power of this Galois group.

One can define quaternionic/octonionic ideles in terms of rational quaternions/octonions multiplied by p-adic number. For adeles this condition produces non-sensical results.

1. This condition indeed allows to construct the inverse of p-adic quaternion/octonion as a product of inverses for rational quaternion/octonion and p-adic number. The reason is that the solutions to $\sum x_i^2 = 0$ involve always p-adic numbers with an infinite number of pinary digits - at least one and the identification excludes this possibility. The ideles form also a group as required.

2. One can interpret also the quaternionicity/octonionicity in terms of Galois group. The 7-dimensional non-associative counterparts for octonionic automorphisms act as transformations $x \rightarrow gxg^{-1}$. Therefore octonions represent this group like structure and the p-adic octonions would have interpretation as combination of octonionic automorphisms with those of rationals.

3. One cannot assign to ideles 4-D idelic surfaces. The reason is that the non-constant part of all 8-coordinates is proportional to the same p-adic valued function of space-time point so that space-time surface would be a disjoint union of effectively 1-dimensional structures labelled by a subset of rational points of M^8. Induced metric would be 1-dimensional and induced Kähler and spinor curvature would vanish identically.

4. One must allow p-adic octonions to have arbitrary p-adic components. The action of ideles representing Galois group on these surfaces is well-defined. Number field property is lost but this feature comes in play as poles only when one considers rational functions. Already the Minkowskian signature forces to consider complexified octonions and quaternions leading to the loss of field property. It would not be surprising if p-adic poles would be associated with the light-like orbits of partonic 2-surfaces. Both p-adic and Minkowskian poles might therefore be highly relevant physically and analogous to the poles of ordinary analytic functions. For instance, n-point functions could have poles at the light-like boundaries of causal diamonds and at light-like partonic orbits and explain their special physical role.

   The action of ideles in the quaternionic tangent space of space-time surface would be analogous to the action of of adelic linear group $Gl_n(A)$ in n-dimensional space.

5. Adelic variants of octonions would be Cartesian products of ordinary and various p-adic octonions and would define a ring. Quaternionic 4-surfaces would define associative local sub-rings of octonion-adelic ring.

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